BPhO 2025 Round 2

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Solutions

1 Question 1 - General Questions

(a) Assume that the chain's horizontal momentum doesn't cause it to overshoot the corner of the table. Also since the rope is stretched out in a straight line, each particle on the rope moves at the same speed at each instant. These together with the fact that table is frictionless make the process elastic (no sudden change in speed of any part of the chain) so that energy is conserved. We assume that the rope has uniform mass distribution so the centre of mass is at the middle point. And we take the level of the table to have height $h = 0$.

Then the initial gravitational potential energy is $U_i = (\frac{1}{2}m)g(0) + (\frac{1}{2}m)g(-\frac{l}{4}) = -\frac{1}{8}mgl$ and the final potential energy is $U_f = mg(-\frac{l}{2}) = -\frac{1}{2}mgl$. Since the rope is released from the rest, conservation of energy gives the kinetic energy

$$\frac{1}{2}mv^2 = \Delta K = -\Delta U = U_i - U_f = \frac{3}{8}mgl \tag{1}$$

which yields

$$v = \sqrt{\frac{3}{4}gl} \tag{2}$$

Substituting $l = 1.0\,\mathrm{m}$ gives $v = 2.71\,\mathrm{m/s}$

(b) As shown in the figure, at a specific height x above the ground, wind exerts a force $\mathrm{d}F = \rho v^2 \mathrm{d}A$ on a semi-circular band of shaded area with width $\mathrm{d}x$. Since wind exerts force on the projected area, the area that stands the wind force of this band is $\mathrm{d}A = 2r\mathrm{d}x$ where r is the radius of the cylinder given by

$$r = \frac{d}{2} = \frac{7.5}{2} = 3.75 \text{ m} \tag{3}$$

The torque exerted by wind force with respect to point O on this band is

$$\mathrm{d}\tau = r'\mathrm{d}F\sin\theta = x\mathrm{d}F = (x)(\rho v^2)(2r\mathrm{d}x) = 2\rho r v^2 x\mathrm{d}x \tag{4}$$

Hence the total torque due to wind force is

$$\tau = \int_0^h 2\rho r v^2 x\mathrm{d}x = \rho r v^2 h^2 \tag{5}$$

Note that this can also be obtained by considering the wind force acting on the height of the centre of mass of the cylinder (middle point). The total force acts on an projected area of $A = (2r)(h) = 2rh$ so the force is

$$F = \rho v^2(2rh) = 2\rho r v^2 h \tag{6}$$

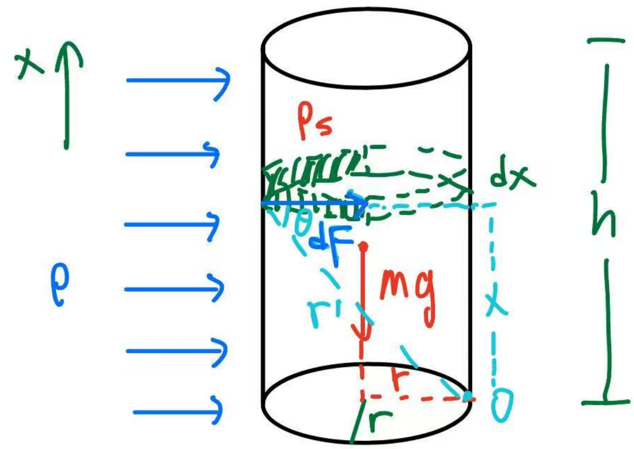

Figure 1: figure for Q1 (b)

and the total torque is

$$\tau = F(\frac{h}{2}) = \rho r v^2 h^2 \tag{7}$$

But this second method only works when the force distribution is constant.

We now consider the balance of torques with respect to point O. The torque due to wind force should equal in magnitude to the torque due to gravity. This gives

$$\rho r v^2 h^2 = Mg(r) = \rho_s(\pi r^2 h)g(r) \tag{8}$$

where ρ_s is the density of the solid tower.

Solving for v gives

$$v = \sqrt{\frac{\pi \rho_s g r^2}{\rho h}} = \sqrt{\frac{(3.14)(3000)(9.8)(3.75)^2}{(1.3)(37)}} = 164 \text{ m/s} \tag{9}$$

above which the tower will topple.

The actual shape of the monument tower has a larger cross sectional area at the base than that at the top. This will increase the portion of the wind force at the base compare to the top and hence decrease the overall wind torque. Also this shape will lower the height of the centre of mass so even though this does not change the initial torque due to gravity, once the tower do topple, the line of action of gravity will be more likely to stay on the original side of O and to provide a restoring torque.

(c)

(i) The angular momentum

$$L = rp = mvr = m(\omega r)r = mr^2\omega = (25 \times 10^{-3})(0.5)^2(2\pi \times 4) = 0.16 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1} \tag{10}$$

(ii) The radius of the wheel is half of the diameter which is $r = 0.31$ m. Condition for rolling without slipping is $v_c = \omega r$, where $v_c = 12$ m/s is the speed of the centre of the circle. Assume the mass $m = 1.3$ kg is uniformly distributed around the circular rim of the wheel, then each particle on the rim with the same differential mass contributes the same angular momentum because radius is constant. Hence following the same reasoning as previous question, the angular momentum with respect to the centre of the wheel is given by

$$L = mr^2\omega = mrv_c = (1.3)(0.31)(12) = 4.8 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1} \tag{11}$$

Note that we assumed, based on the naivety shown in the description of this question, that it asks for the angular momentum with respect to the centre of the wheel. If, instead, it asks for the angular momentum with respect to a fixed point on the ground, then based on the König's Theorem, we have to add the angular momentum of the centre of mass itself, which is another mrv_c. So the final result would be twice the above.

(iii) Let's consider the i-th particle. Its distance to the origin is $\frac{i}{N}l$ and mass is $\frac{M}{N}$. So the total angular momentum is given by the sum of the angular momentum for individual particles, that is

$$\begin{aligned} L &= \sum_{i=1}^{N}(\frac{M}{N})(\frac{i}{N}l)^2\omega = \frac{Ml^2\omega}{N^3}\sum_{i=1}^{N}i^2 \\ &= \frac{N(N+1)(2N+1)}{6N^3}Ml^2\omega \end{aligned} \tag{12}$$

(iv) We take the limit when $N \to \infty$, the angular momentum becomes

$$L = \frac{1}{3}Ml^2\omega \tag{13}$$

(v) The speed of i-th particle is $v_i = \omega\frac{il}{N}$, so the kinetic energy is

$$\begin{aligned} K &= \sum_{i=1}^{N}\frac{1}{2}(\frac{M}{N})v_i^2 = \sum_{i=1}^{N}\frac{1}{2}(\frac{M}{N})\omega^2\frac{i^2l^2}{N^2} \\ &= \frac{M\omega^2l^2}{2N^3}\sum_{i=1}^{N}i^2 = \frac{M\omega^2l^2}{2N^3}\frac{N(N+1)(2N+1)}{6} \end{aligned} \tag{14}$$

As $N \to \infty$, we have

$$K = \frac{1}{2}(\frac{1}{3}Ml^2)\omega^2 \tag{15}$$

2 Question 2 - Random Walks

(a) Average translational kinetic energy is

$$\frac{1}{2}mv^2 = \frac{3}{2}k_BT \tag{16}$$

where v is the centre of mass velocity of N_2 gas molecules. So root-mean-square velocity is

$$v_{\text{rms}} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3N_Ak_BT}{N_Am}} = \sqrt{\frac{3RT}{M_R}} \tag{17}$$

where, N_A is the Avogadro's constant, $M_R = 0.028\ \text{kg} \cdot \text{mol}^{-1}$ is the molar mass of N_2, and $R = 8.31\ \text{J} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$ is the molar gas constant. Using typical room temperature 293 K, we have

$$v_{\text{rms}} = 511\ \text{m/s} \tag{18}$$

(b) From the definition of density

$$\rho = \frac{M}{V} = \frac{M_R \cdot n}{V} = \frac{M_R(\frac{N}{N_A})}{V} = \frac{M_RN}{N_AV} \tag{19}$$

as required, where n is the number of moles and M is the total mass.

The average volume occupied by each particle is

$$\bar{\text{v}} = \frac{V}{N} = \frac{M_R}{\rho N_A} \tag{20}$$

(c) Assume the liquid N_2 molecule is a hard sphere of diameter d and the molecules pack closely in the liquid phase, we have

$$\frac{\pi}{6}d^3 = \frac{M_R}{\rho N_A} \tag{21}$$

so

$$d = \sqrt[3]{\frac{6M_R}{\pi\rho N_A}} = 4.8 \times 10^{-10}\ \text{m} \tag{22}$$

(d) Note that in this question we come back to N_2 gas so cannot use the ρ of liquid nitrogen, but the diameter of molecules is the same. We can also use the root-mean-square velocity in (a) as it is for the gas phase. The mean-free-path is

$$\lambda = \frac{V}{\sqrt{2}\pi d^2 N} = \frac{1}{\sqrt{2}\pi d^2} \cdot \frac{k_BT}{p} = 3.9 \times 10^{-8}\ \text{m} \tag{23}$$

where we have used the ideal gas equation $pV = Nk_BT$

The typical collision time is given by

$$\Delta t = \frac{\lambda}{v_{\text{rms}}} = \frac{3.9 \times 10^{-8}}{511} = 7.6 \times 10^{-11}\ \text{s} \tag{24}$$

(e) the velocity is

$$\boldsymbol{v}_i = v\cos\theta_i\hat{\mathbf{x}} + v\sin\theta_i\hat{\mathbf{y}} \tag{25}$$

(f) From the assumption that the time between collisions Δt is constant from one collision to the next, and the approximation that the speed v of any given molecule is constant in time between collisions, the displacement $\boldsymbol{r}_i$ between each collision is thus $\boldsymbol{r}_i = \boldsymbol{v}_i \Delta t$. So the vector displacement

$$\boldsymbol{r}(t) = \sum_{i=1}^{M} \boldsymbol{r}_i = \sum_{i=1}^{M} \boldsymbol{v}_i \Delta t \tag{26}$$

(g) The average vector displacement

$$\langle \boldsymbol{r}(t) \rangle = \langle \sum_{i=1}^{M} v \Delta t \cos\theta_i \hat{\mathbf{x}} + v \Delta t \sin\theta_i \hat{\mathbf{y}} \rangle = v\Delta t \langle \sum_{i=1}^{M} \cos\theta_i \rangle \hat{\mathbf{x}} + v\Delta t \langle \sum_{i=1}^{M} \sin\theta_i \rangle \hat{\mathbf{y}} \tag{27}$$

$$= v\Delta t M \langle \cos\theta \rangle \hat{\mathbf{x}} + v\Delta t M \langle \sin\theta \rangle \hat{\mathbf{y}} \tag{28}$$

and because of the random nature of the motion both $\langle \cos\theta \rangle$ and $\langle \sin\theta \rangle$ average to 0. Hence we have $\langle \boldsymbol{r}(t) \rangle = 0$

(h) The square of the expected distance is

$$\langle |\boldsymbol{r}|^2 \rangle = \langle \boldsymbol{r} \cdot \boldsymbol{r} \rangle = v^2 \Delta t^2 \sum_{i=1}^{M} \sum_{j=1}^{M} (\langle \cos\theta_i \cos\theta_j \rangle + \langle \sin\theta_i \sin\theta_j \rangle) \tag{29}$$

where we have used $\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = 1$ and $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$

From the random nature of the motion we know

$$\sum_{i \neq j}^{M} \langle \cos\theta_i \cos\theta_j \rangle = \sum_{i \neq j}^{M} \langle \cos\theta_i \rangle \langle \cos\theta_j \rangle = 0 \tag{30}$$

and similarly

$$\sum_{i \neq j}^{M} \langle \sin\theta_i \sin\theta_j \rangle = 0 \tag{31}$$

because when $i \neq j$ the two angles are independent variables so average can be taken separately in a product. But we need to consider the non-zero contribution from $i = j$, which gives us

$$\sum_{i=1}^{M} \sum_{j=1}^{M} (\langle \cos\theta_i \cos\theta_j \rangle + \langle \sin\theta_i \sin\theta_j \rangle) = \sum_{i=1}^{M} \langle \cos^2\theta_i + \sin^2\theta_i \rangle = \sum_{i=1}^{M} 1 = M \tag{32}$$

Hence we have

$$\langle |\boldsymbol{r}|^2 \rangle = r^2 \Delta t^2 M \tag{33}$$

and this leads to

$$\langle |\boldsymbol{r}| \rangle = v \Delta t \sqrt{M} \tag{34}$$

(i) As required we hold the assumption that in 3D the distance $r = v\Delta t\sqrt{M}$. The total time taken for diffusion should be $t = M\Delta t$. So we have

$$M = \frac{t}{\Delta t} \tag{35}$$

and hence

$$r = v\Delta t\sqrt{M} = v\Delta t\sqrt{\frac{t}{\Delta t}} = v\sqrt{t\Delta t} \tag{36}$$

which gives

$$t = \frac{r^2}{v^2\Delta t} \tag{37}$$

The temperature and pressure conditions are the same as in (d) so we can use the Δt value there. For the velocity v we use the v_{rms} as before. Finally substituting $r = 5.00$ m we have

$$t = 1.26 \times 10^6 \text{ s} \tag{38}$$

3 Question 3 - The Blast Radius of The Gadget

(a) The relevant units are $[\rho] = ML^{-3}$, $[t] = T$, $[E] = MLT^{-2}$, and $[R] = L$. Since $R \propto \rho^\alpha t^\beta E^\gamma$, we have

$$L = (ML^{-3})^\alpha (T)^\beta (MLT^{-2})^\gamma = M^{\alpha+\gamma} T^{\beta-2\gamma} L^{-3\alpha+\gamma} \tag{39}$$

which gives us the system of equations

$$\begin{cases} \alpha + \gamma = 0 \\ \beta - 2\gamma = 0 \\ -3\alpha + \gamma = 1 \end{cases} \tag{40}$$

and the solution is

$$\begin{cases} \alpha = -\frac{1}{5} \\ \gamma = \frac{1}{5} \\ \beta = \frac{2}{5} \end{cases} \tag{41}$$

So if we assume the dimensionless constant $k = 1$ we have the equation

$$R = \left(\frac{Et^2}{\rho}\right)^{\frac{1}{5}} \tag{42}$$

(b) Assume ideal gas $pV = nRT$ where n is the number of moles. Hence the pressure

$$p = \frac{n}{V} RT = \frac{M/M_R}{M/\rho} RT = \frac{\rho}{M_R} RT \tag{43}$$

where M is the total mass and we used $n = \frac{M}{M_R}$ and $\rho = \frac{M}{V}$
Rearrange gives the equation

$$\rho = \frac{M_R p}{RT} \tag{44}$$

Typical sea level pressure is $p = 1.01 \times 10^5$ Pa and temperature is $T = 280$ K. We can use the main component of air, nitrogen, to estimate the molar mass of air, which is roughly $M_R = 0.029 \text{ kg} \cdot \text{mol}^{-1}$. Substituting all these numbers into the equation we have

$$\rho = 1.26 \approx 1.3 \text{ kg} \cdot \text{m}^{-3} \tag{45}$$

as required

(c) The speed of sound is

$$c_s = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{1.4 \times 1.01 \times 10^5}{1.3}} = 330 \text{ m/s} \tag{46}$$

(d) From the figure $R \sim 150$ m, and we know $t = 0.025$ s. The energy released is given by

$$E = \frac{\rho R^5}{t^2} = \frac{1.3 \times 150^5}{0.025^2} = 1.6 \times 10^{14} \text{ J} = 160 \text{ TJ} \approx 100 \text{ TJ} \tag{47}$$

(e) Assume 100 TJ of energy is constant, substituting into the equation we get 37.8, 94.9, and 238 meters respectively.

(f) For constant E, we have shock front speed

$$v_s = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{5}\left(\frac{E}{\rho}\right)^{\frac{1}{5}} t^{-\frac{3}{5}} \tag{48}$$

Substituting into this equation we get 15110, 3795, 953 m/s respectively.

(g) Once the flow behind the shock becomes subsonic, sound waves can outrun the front and radiate energy away into the atmosphere as audible noise and infrasound. The remaining ordered kinetic energy converts into chaotic turbulent motion and heat through internal friction, which is far more efficient at subsonic speeds. Ambient pressure can now send signals into the blast wave, doing work against its expansion and sapping its energy.

(h) Let $v_s = c_s$, the time of stall is t_s is such that

$$c_s = \frac{2}{5}\left(\frac{E}{\rho}\right)^{\frac{1}{5}} t_s^{-\frac{3}{5}} \tag{49}$$

so

$$t_s = \sqrt[3]{\frac{E}{\rho(\frac{5}{2}c_s)^5}} = 0.59 \text{ s} \tag{50}$$

and substituting this into the radius equation derived in (a) gives

$$R = 480 \text{ m} \tag{51}$$

(i) Secondary effects such as flying debris and sound waves travel far beyond the stalled shock.

Note that this question shares very similar ideas with Question 2 of the 2021 PUPC competition.

4 Question 4 - The Pendulum

(a) As shown in the figure, we set the level of the pivot as zero height.

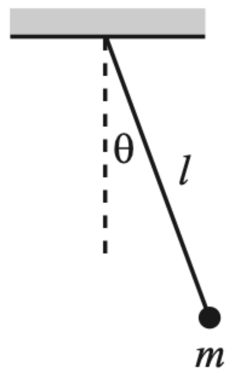

Figure 2: figure for Q4

The kinetic energy of the mass is $K = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$, and the potential energy is $U = mgh = -mgl\cos\theta$. Since there is no work done due to non-conservative forces, the total mechanic energy $E = K + U$ is conserved. Hence

$$E = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \tag{52}$$

is conserved. This means that

$$0 = \dot{E} = ml^2\dot{\theta}\ddot{\theta} + mgl\sin\theta\dot{\theta} \tag{53}$$

which can be simplified to

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{54}$$

Now we state the assumption we make, which is the amplitude θ_0 is small (less than 10 degrees for instance). For small angles, we have the approximation $\sin\theta \approx \theta$. Also define

$$\omega = \sqrt{\frac{g}{l}} \tag{55}$$

the differential equation now becomes

$$\ddot{\theta} + \omega^2\theta = 0 \tag{56}$$

This has general solution

$$\theta(t) = A\cos(\omega t + \psi) \tag{57}$$

for arbitrary constants A and ψ. The period of this function is then

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \tag{58}$$

as required.

(b) When the mass is at the highest position during oscillation, its velocity and hence kinetic energy is 0. So the total mechanical energy is only due to the potential energy and we have

$$E = -mgl\cos\theta_0 \tag{59}$$

Thus

$$\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = -mgl\cos\theta_0 \tag{60}$$

and rearrange gives

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0) \tag{61}$$

(c) Taking the square root gives

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_0} \tag{62}$$

so

$$\mathrm{d}t = \sqrt{\frac{l}{2g}}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{63}$$

Due to the symmetry of the oscillation, moving from angle 0 to θ_0 is one-fourth of the full period. So the total period is

$$T = 4\int_0^{\theta_0}\mathrm{d}t = 4\int_0^{\theta_0}\sqrt{\frac{l}{2g}}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \sqrt{\frac{8l}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{64}$$

(d) Differentiate both sides of the substitution condition

$$\sin\phi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}} \tag{65}$$

gives

$$\cos\phi\mathrm{d}\phi = \frac{\cos\frac{\theta}{2}\mathrm{d}\theta}{2\sin\frac{\theta_0}{2}} \tag{66}$$

this gives

$$\mathrm{d}\theta = \frac{2\sin\frac{\theta_0}{2}\cos\phi\mathrm{d}\phi}{\cos\frac{\theta}{2}} \tag{67}$$

Using the identity $\cos x = 1 - 2\sin^2\frac{x}{2}$, the period expression can be simplified to

$$\begin{aligned} T &= \sqrt{\frac{8l}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} \\ &= \sqrt{\frac{8l}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{(1-2\sin^2\frac{\theta}{2}) - (1-2\sin^2\frac{\theta_0}{2})}} \\ &= \sqrt{\frac{4l}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}} \end{aligned} \tag{68}$$

And use the substitution condition gives

$$
\begin{aligned}
T &= 2\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{\mathrm{d}\theta}{\sin\frac{\theta_0}{2}\sqrt{1-\sin^2\phi}} = 2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{\frac{2\cancel{\sin\frac{\theta_0}{2}}\cancel{\cos\phi}\mathrm{d}\phi}{\cos\frac{\theta}{2}}}{\cancel{\sin\frac{\theta_0}{2}}\cancel{\cos\phi}} \\
&= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\cos\frac{\theta}{2}} = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{1-\sin^2\frac{\theta}{2}}} \\
&= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\phi}}
\end{aligned} \tag{69}
$$

where the change of integration limit is correct because when $\theta = \theta_0$, we have $\sin\phi = 1$ which means $\phi = \frac{\pi}{2}$

We now define

$$
k = \sin\frac{\theta_0}{2} \tag{70}
$$

so that

$$
T = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{71}
$$

(e) When θ_0 is very small, $\sin\frac{\theta_0}{2}\approx\frac{\theta_0}{2}$. And using the binomial approximation we have

$$
\begin{aligned}
T &= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\phi}} = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}(1-\frac{\theta_0^2}{4}\sin^2\phi)^{-\frac{1}{2}}\mathrm{d}\phi \\
&\approx 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}1+\frac{\theta_0^2}{8}\sin^2\phi\,\mathrm{d}\phi = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_0^2}{4\pi}\int_0^{\frac{\pi}{2}}\sin^2\phi\,\mathrm{d}\phi\right) \\
&= 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_0^2}{4\pi}\cdot\frac{\pi}{4}\right) = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_0^2}{16}\right)
\end{aligned} \tag{72}
$$

This is a second order correction to the period we got in (a). If we ignore higher than first order corrections due to small θ_0, i.e. set $k = 0$, we recover the previous period.

(f) The mass released from the same level of the pivot means $\theta_0 = \frac{\pi}{2}$ so $\cos\frac{\theta_0}{2} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. The small angle period $\tau = 2\pi\sqrt{\frac{1.00}{9.8}} = 2.00709$ s. Now we focus on the agm function agm$[\frac{\sqrt{2}}{2}, 1]$. We start with

$$
a_0 = \frac{\sqrt{2}}{2} = 0.7071 \ , \quad b_0 = 1 \tag{73}
$$

then

$$
a_1 = 0.8536 \ , \quad b_1 = 0.8409 \tag{74}
$$

and

$$
a_2 = 0.8472 \ , \quad b_2 = 0.8472 \tag{75}
$$

which is a good enough approximation. The period is

$$
T = \frac{2.00709}{0.8472} = 2.37 \text{ s} \tag{76}
$$

to 3 significant figures.

(g) For large angle θ_0, we repeat the derivation using conservation of total mechanical energy. The total energy is

$$\begin{aligned} E &= \frac{1}{2}(\frac{m}{2})l^2\dot{\theta}^2(1+\alpha)^2 + \frac{1}{2}(\frac{m}{2})l^2\dot{\theta}^2(1-\alpha)^2 - (\frac{m}{2})gl(1+\alpha)\cos\theta - (\frac{m}{2})gl(1-\alpha)\cos\theta \\ &= \frac{1}{2}ml^2\dot{\theta}^2(1+\alpha^2) - mgl\cos\theta \end{aligned} \tag{77}$$

and we also know that the total energy equals to $-(\frac{m}{2})gl(1+\alpha)\cos\theta_0 - (\frac{m}{2})gl(1-\alpha)\cos\theta_0$ which is $-mgl\cos\theta_0$ from the highest point. Hence

$$\frac{1}{2}ml^2\dot{\theta}^2(1+\alpha^2) - mgl\cos\theta = -mgl\cos\theta_0 \tag{78}$$

Following the exact same process as before we can derive

$$\dot{\theta}^2 = \frac{2g}{l(1+\alpha^2)}(\cos\theta - \cos\theta_0) \tag{79}$$

and similarly the overall period

$$T' = 4\int_0^{\theta_0} \mathrm{d}t = \sqrt{\frac{8l(1+\alpha^2)}{g}} \int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{80}$$

Compare the periods we notice

$$T' = \sqrt{1+\alpha^2}T \tag{81}$$

and since $0 < \alpha < 1$ we conclude that $T' > T$, the period is longer than simple pendulum.

Note that same result can be obtained in the regime of small angle approximation, in which case we use the formula of physical pendulum

$$T' = 2\pi\sqrt{\frac{I}{Mgd}} \tag{82}$$

where the total mass $M = 2 \cdot \frac{m}{2} = m$, the momentum of inertia should be

$$I = (\frac{m}{2})l^2(1+\alpha)^2 + (\frac{m}{2})l^2(1-\alpha)^2 = ml^2(1+\alpha^2) \tag{83}$$

and the distance to centre of mass $d = l$. So

$$T' = 2\pi\sqrt{\frac{ml^2(1+\alpha^2)}{mgl}} = 2\pi\sqrt{\frac{l(1+\alpha^2)}{g}} \tag{84}$$

which is still $\sqrt{1+\alpha^2}$ times the simple case.