

A13481W1

SECOND PUBLIC EXAMINATION

Honour School of Physics – Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B2. SYMMETRY AND RELATIVITY

TRINITY TERM 2016

Wednesday, 15 June, 2.30 pm – 4.30 pm

Answer five questions with at least one from each section:

Start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

The following notation is used throughout the paper: capital bold letters (e.g. $\mathbf{U}$) indicate 4-vectors; lower case bold letters (e.g. $\mathbf{v}$) indicate 3-vectors. Note, however, that the symbols $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{A}$ in questions 3 and 4 indicate 3-vectors: the electric and magnetic fields and vector potential respectively.

1. For a particle of mass m moving along a world-line in an inertial reference frame, S , define the proper time, τ , 4-velocity, $\mathbf{U}$ and 4-acceleration, $\mathbf{A}$. Using the relation for the γ -factor, $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$ of the particle, $d\gamma/dt = \gamma^3(\mathbf{v} \cdot \mathbf{a}/c^2)$, where $\mathbf{v}$ and $\mathbf{a}$ are the particle 3-velocity and 3-acceleration, correspondingly, find $\mathbf{A}$ in terms of γ , $\mathbf{v}$ and $\mathbf{a}$.

Find the invariants of 4-velocity and 4-momentum. Evaluate explicitly the scalar product, $\mathbf{U} \cdot \mathbf{A}$. [5]

Two events in S are characterized by 4-coordinates, $\mathbf{D} = (ct_d, \mathbf{x}_d)$ and $\mathbf{B} = (ct_b, \mathbf{x}_b)$, where $\mathbf{x}_d$ and $\mathbf{x}_b$ are 3-vectors. Write down the condition for these events to be connected by a time-like interval. In such a case, can we find an inertial frame, S' , where the two events are occurring simultaneously? Explain. What is the physical meaning of the condition $\mathbf{D} \cdot \mathbf{B} = 0$? [5]

Define proper acceleration and pure force. A particle undergoing hyperbolic motion has a worldline given by $x^2 - t^2 = L^2$, where L is a constant and the speed of light is $c = 1$. Find the particle's speed, v , the Lorentz factor, $\gamma(v)$ and the acceleration, a , as functions of x and show that $a\gamma^3$ is constant. [5]

Now consider the motion of a particle of mass m , initially at rest under the influence of a constant 3-force, $\mathbf{f}$. Find $\beta \equiv \mathbf{v}/c$ and the Lorentz factor $\gamma(t)$ of the particle as a function of time t . Sketch the graphs of $\gamma(t)$ and $\beta(t)$. [5]

An electron is accelerated from rest through a gap of $L = 10$ m by an electric field of strength 5 MV m^{-1} that is constant throughout the gap. Find γ and β at the other end of the gap. How much would β change if the accelerating field was reduced by 20%? How long does it take for the electron to reach the other end of the gap? [5]

$$P_{\perp} = eBR$$

$$P_{\parallel} = \gamma m v_z$$

$$P_{\perp} = \gamma m v_{\perp}$$

$$P = \gamma m \underline{v}$$

$$\gamma m v_z = eBR$$

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$$R = \frac{\gamma m v_z}{eB} \text{ at } t = t_0$$

2. A frame S' is moving relative to the laboratory frame, S , with velocity $\underline{v} = (v_x, 0, 0)$. A particle of mass m moves in S with velocity $\underline{u} = (u_x, u_y, 0)$. Let θ be the angle between $\underline{u}$ and $\underline{v}$ in S . Using $\underline{u} = \underline{u}_{\parallel} + \underline{u}_{\perp}$, where $\underline{u}_{\parallel}$ is the component of the particle's velocity in the direction of motion of S' and $\underline{u}_{\perp}$ is the component perpendicular to it, show that the angle θ' in frame S' is given by

$$\tan \theta' = \frac{u \sin \theta}{\gamma_v (u \cos \theta - v)}, \quad \frac{dP}{dt}$$

where $\gamma_v \equiv \gamma(v)$ is the Lorentz factor.

[5]

In frame S an electron moves in a uniform magnetic field, $\underline{B} = (0, 0, B)$, along a spiral trajectory defined by the Larmor radius, R , and a constant longitudinal velocity, $v_z \parallel \underline{B}$. The initial 4-momentum of the electron is $\underline{P}_0$. The ratio between longitudinal and transverse components of the electron's 3-momentum is $p_{\parallel}/p_{\perp} = 1$, while the electron's Lorentz factor is $\gamma = 17$. At $t = t_0$ a constant electric field, $\underline{E} = (0, 0, E_z)$, is applied in such a way that it decelerates the electron. After propagating 1 m along the z direction, the electron's longitudinal velocity drops to zero, $v_z = 0$. Find the strength, E_z , of the electric field applied and the factor γ at the end of the trajectory. Sketch the dependence of the Larmor radius on z . Ignore any radiation effects by the electron.

[7]

[Hint: Use the fact that the electron's rest energy in eV is $m_0 c^2 = 0.511$ MeV and $\gamma = 1 + eV/m_0 c^2$, where V is the voltage applied.]

Two photons of the same angular frequency, ω , and with 4-momenta $\underline{P}_1$ and $\underline{P}_2$, move in the stationary frame S . The first photon moves along the x direction and has $v_y = 0$ while the second photon moves at some angle θ to the x direction. Both photons have $v_z = 0$. Find:

- the rest energy of the system as a function of ω and θ ; and
- the velocity of the centre of mass frame relative to the lab frame as a function of θ , making a sketch of the θ -dependence.

$$\left(\underline{f} = \gamma m_0 \underline{a} + \frac{\underline{f} \cdot \underline{u}}{c^2} \underline{u} \right) \quad \gamma m_0 \underline{a} = \underline{f} - \frac{\underline{f} \cdot \underline{u}}{c^2} \underline{u}$$

[7]

Photons with wave 4-vector $\underline{K}$ are radiated by a stationary laser towards a beam of electrons. Each electron in the beam has energy W and velocity v along the x direction. A single photon scatters off an electron. In the lab frame, S , find the maximum photon energy after the scattering. If the electron's energy is 2 GeV and the photon's wavelength is 1 cm, calculate the scattered photon's wavelength.

[6]

$$\gamma_z = \gamma_0 - \frac{e z E_z}{m_0 c^2}$$

$$\gamma_z = \gamma_0 - \frac{e z E_z}{m_0 c^2}$$

$$\frac{1}{\gamma_f} = \frac{1}{\gamma_i} \frac{P_z = \gamma_i m_0 v}{P_{\parallel}(z) = e B R z}$$

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$$\frac{d\gamma}{dt} = \frac{\underline{f} \cdot \underline{u}}{m_0 c} = \frac{e E_z}{m_0 c^2} u_z$$

$$E_i + eV = E_f$$

$$\gamma_i m_0 c^2 + eV = \gamma_f m_0 c^2$$

$$\gamma_f = \gamma_i + \frac{eV}{m_0 c^2}$$

[Turn over]

$$\frac{eV}{m_0 c^2} = -16$$

$$V = -\frac{16 m_0 c^2}{e} = (1 \text{ m}) \times E_z$$

$$k = \frac{\omega n}{c}$$

3. Name three phenomena, each of which can be detected via the studies of emitted or reflected light in astrophysics, that would represent a direct test of Special Relativity. Explain what the relativistic Doppler effect is and derive the equation for the photon's frequency shift from ω to ω' between reference frames S and S', respectively, moving with relative velocity $\mathbf{v}$. The photon is emitted at an angle θ with respect to $\mathbf{v}$ in S. [6]

A plane monochromatic electromagnetic wave with angular frequency ω and wave 3-vector $\mathbf{k}$ propagates in a uniform medium with refractive index n . Define the phase and group velocities, v_{ph} and v_{gr} , and show that $v_{gr}v_{ph} = c^2$ for $n = 1$. [4]

An electron of mass m_e and a proton of mass M_p are moving in the lab frame S in opposite directions but toward each other with velocities, $\mathbf{v} = (0, 0, \pm v_z)$, respectively. At some moment the particles collide and a photon is emitted in the direction perpendicular to the z axis. After the collision, the electron and the proton are moving as a single particle. The wavelength of the photon as measured by an observer in the laboratory frame is λ . Find the minimal total energy of the electron and proton, E_{tot} , before the collision that would allow emission of such a photon. [7]

A plane, linearly polarized electromagnetic wave propagates in the z direction through a uniform medium with the refractive index $n > 1$. The electric and magnetic fields of the wave are given by

$$E_y = E_0 \cos(\omega t - kz), \quad B_x = B_0 \cos(\omega t - kz) \quad \text{and} \quad |\mathbf{E}_0 \times \mathbf{B}_0| = 1.$$

Find the vector potential, $\mathbf{A}$, the scalar potential, φ and the wave 4-vector, $\mathbf{K}$, in the Lorentz gauge, $\partial^\mu A_\mu = 0$. Is it possible to find a reference frame in which either the electric or magnetic field of the wave defined above would vanish? If yes, find the frame's velocity relative to the lab frame. [8]

$$E_{tot} = E + E_m$$

$$E_p = \frac{m_p}{m_e} E_e$$

$$E_e + E_p = E + E_m$$

$$p_p - p_e = p_m \cos \theta$$

$$E = p_m \sin \theta$$

$$E_{tot} = \left(1 + \frac{m_p}{m_e}\right) E_e$$

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 \hat{y} \operatorname{Re}(\exp(i(\omega t - kz)))$$

$$-M^2 = -(E_e + E_p)^2 + (p_p - p_e)^2$$

$$-m_e^2 - m_p^2 + 2p_e \cdot p_p$$

$$-2E_e E_p - 2p_e p_p$$

$$-\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

4. Write down the relation between the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, and the components of the 4-vector potential, A^μ . Define the field strength tensor, $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ and show that its components satisfy $\partial^c F^{ab} + \partial^a F^{bc} + \partial^b F^{ca} = 0$. Define the 4-current, J_ν , and show that two out of the four Maxwell's equations can be written in the form $\partial^\mu F_{\mu\nu} = J_\nu$.

Show that the Lorentz force acting on the unit volume of charge density, ρ , can be written as $f_\mu = J^\nu F_{\nu\mu}$. What is the physical meaning of the f_0 component of this 4-vector?

Consider a unit vector, $U_W^\mu = W^\mu / (m_0 c |\mathbf{s}|)$, parallel to the Pauli-Lubanski spin vector, $W^\mu = (\mathbf{s} \cdot \mathbf{p}, E\mathbf{s}/c)$. Show without direct calculations that U_W^μ and the 4-momentum are orthogonal to each other, i.e. $U_W^\mu P_\mu = 0$. [Hint: Use the rest frame.]

A free relativistic electron (with $\gamma = 100$) moves (a) in a vacuum (refractive index $n = 1$), or (b) in a uniform medium (refractive index $n = 2$). Explain whether the electron emits a photon in either of the cases listed. If the electron in case (a) or case (b) can emit a photon, at what angle with respect to the electron's trajectory is the photon emitted?

A straight wire with the charge density ρ moves with velocity $\mathbf{v}$ along the z axis. Find the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, generated by the wire in the frames:

- S' which is co-moving with the wire;
- the laboratory (stationary) frame S .

A second identical wire having the same charge density and separated by the distance, d , from the first wire moves parallel to the first wire with the same velocity. Is it possible to find such a frame that the forces between wires can be described as purely magnetic? Prove the statement.

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\frac{\partial}{\partial x} \mathcal{P}$$

→

$$c = \frac{\omega}{k} = \lambda \nu$$

$$\nu \rightarrow \frac{c}{\lambda} \quad \lambda \rightarrow \frac{c}{\nu} \quad \nu \rightarrow \nu$$

$$\frac{c h}{\lambda} \quad h \nu \rightarrow h \nu$$

$$\omega = \lambda \nu$$

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- proper time T is the time experienced by an observer in its own rest frame. ✓

4-velocity $U = \frac{dx}{dT}$, where $X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$ is
 $= \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$

the position 4-vector

4-acceleration $A = \frac{dU}{dT}$

$$\frac{dT}{dt} = \gamma, \quad \frac{d\gamma}{dt} = \gamma^3 (\underline{v} \cdot \underline{a}) \frac{1}{c^2}$$

$$\begin{aligned} \therefore A = \frac{dU}{dT} &= \frac{d}{dT} \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix} = \frac{d}{dt} \frac{dT}{dT} \frac{d}{dT} \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix} \\ &= \gamma \begin{pmatrix} \frac{d}{dt} (\gamma c) \\ \frac{d}{dt} (\gamma \underline{v}) \end{pmatrix} \end{aligned}$$

$$= \gamma \begin{pmatrix} c \frac{d\gamma}{dt} \\ \underline{v} \frac{d\gamma}{dt} + \gamma \frac{d\underline{v}}{dt} \end{pmatrix}$$

$$= \gamma \begin{pmatrix} c \frac{\gamma^3}{c^2} (\underline{v} \cdot \underline{a}) \\ \underline{v} \frac{\gamma^3}{c^2} (\underline{v} \cdot \underline{a}) + \gamma \underline{a} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\gamma^4}{c} (\underline{v} \cdot \underline{a}) \\ \frac{\gamma^4}{c^2} (\underline{v} \cdot \underline{a}) \underline{v} + \gamma^2 \underline{a} \end{pmatrix}$$

- invariant

$$\begin{array}{ccc} U & , & P = m_0 U \\ \uparrow & & \uparrow \\ \text{4 velocity} & & \text{4 momentum} \end{array}$$

$$P \cdot U = m_0 U \cdot U = -m_0 c^2$$

$$(U \cdot U = \begin{pmatrix} c \\ 0 \end{pmatrix} \cdot \begin{pmatrix} c \\ 0 \end{pmatrix} = -c^2)$$

$\uparrow$ in rest frame

$$P \cdot P = -m_0^2 c^2$$

$$U \cdot A = \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix} \cdot \begin{pmatrix} \frac{\gamma^4}{c^2} (\underline{v} \cdot \underline{a}) \\ \frac{\gamma^4}{c^2} (\underline{v} \cdot \underline{a}) \underline{v} + \gamma^2 \underline{a} \end{pmatrix}$$

$$= \cancel{-\gamma^5 c^2} - \gamma^5 (\underline{v} \cdot \underline{a}) + \frac{\gamma^5}{c^2} (\underline{v} \cdot \underline{a}) v^2 + \gamma^3 \underline{v} \cdot \underline{a}$$

$$= -\gamma^5 (\underline{v} \cdot \underline{a}) + \gamma^5 \underline{v} \cdot \underline{a} \left(\frac{v^2}{c^2} + \frac{1}{\gamma^2} \right)$$

$$\Rightarrow = -\gamma^5 (\underline{v} \cdot \underline{a}) + \gamma^5 \underline{v} \cdot \underline{a} \left(\frac{v^2}{c^2} + \underbrace{1 - \frac{v^2}{c^2}}_1 \right)$$

$$= -\gamma^5 (\underline{v} \cdot \underline{a}) + \gamma^5 (\underline{v} \cdot \underline{a})$$

$$= \underline{\underline{0}}$$

$$\therefore \underline{U \cdot A = 0}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\underline{D=B}$$

interval $\Delta S^2 = (D-B) \cdot (D-B)$

$$= -c^2(t_d - t_b)^2 + |\underline{x}_d - \underline{x}_b|^2$$

If time-like $\Delta S^2 < 0$

$$\therefore \underline{c^2(t_d - t_b)^2 > |\underline{x}_d - \underline{x}_b|^2}$$

Consider $D = \begin{pmatrix} t_d \\ x_d \\ 0 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} t_b \\ x_b \\ 0 \\ 0 \end{pmatrix}$ in S

If in S' they are simultaneous, then

$$t_d' = t_b'$$

$$t_d' = \gamma(t_d - \frac{Vx_d}{c^2})$$

$$t_b' = \gamma(t_b - \frac{Vx_b}{c^2})$$

$$\therefore t_d - \frac{Vx_d}{c^2} = t_b - \frac{Vx_b}{c^2}$$

$$\therefore c^2(t_d - t_b) = \frac{V}{1} (x_d - x_b)$$

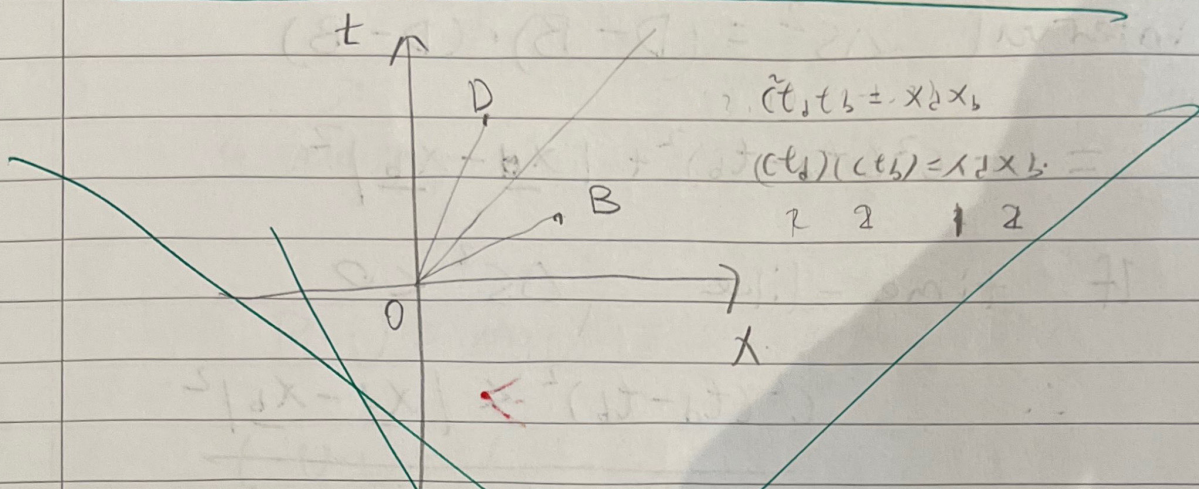
$$\therefore |V| = \frac{|t_d - t_b|}{|x_d - x_b|} c^2$$

$\therefore$ time like $\therefore c^2|t_d - t_b|^2 > |x_d - x_b|^2$

$$\therefore \frac{|t_d - t_b|}{|x_d - x_b|} > \frac{1}{c}$$

$$\therefore |V| > \frac{1}{c} \cdot c^2 = c$$

∴ This is impossible



$$D \cdot B = 0 \rightarrow (ct_d)(ct_b) = |x_d| |x_b| \cos \alpha$$

$$\therefore \frac{|x_d|}{ct_d} \geq \frac{ct_b}{|x_b|} \quad \cos \alpha = \frac{(ct_d)(ct_b)}{|x_d| |x_b|} \leq 1$$

— OD and OB are symmetric about the 45° photon worldline

So OB is the line of simultaneity for an observer whose worldline is OD, and vice versa

* If B is space like, D is time like curve versa

— proper acceleration is the acceleration of an object measured by an observer instantaneously at rest with respect to the object.

— pure force is the force that does not change the rest mass of an object.

World line is $x^2 - t^2 = L^2 \rightarrow \cancel{t = \sqrt{x^2 - L^2}}$

$$V = \frac{dx}{dt} \quad \therefore \cancel{2x} \frac{dx}{dt} - 2t = 0$$

$$\therefore \frac{dx}{dt} = \frac{t}{x} = \frac{t}{\sqrt{x^2 - L^2}}$$

$$x = \sqrt{L^2 - t^2}$$

$$\therefore V = \frac{dx}{dt} = \frac{t}{x}$$

$$= \frac{t}{\sqrt{L^2 - t^2}}$$

$$\gamma(V) = \frac{1}{\sqrt{1 - V^2}} = \frac{1}{\sqrt{1 - \frac{t^2}{L^2 - t^2}}} = \frac{1}{\sqrt{\frac{L^2 - t^2 - t^2}{L^2 - t^2}}}$$

$$t = \sqrt{x^2 - L^2}$$

$$\therefore V = \frac{\sqrt{x^2 - L^2}}{x}$$

$$\gamma(V) = \frac{1}{\sqrt{1 - V^2}} = \frac{1}{\sqrt{1 - \frac{x^2 - L^2}{x^2}}} = \frac{1}{\sqrt{\frac{x^2 - x^2 + L^2}{x^2}}} = \frac{x}{L}$$

$$\therefore x \frac{dx}{dt} - t = 0 \quad \therefore \frac{dx}{dt} \cdot \frac{dx}{dt} + x \frac{d^2x}{dt^2} - 1 = 0$$

$$\therefore V^2 + xa = 1 \quad a = \frac{1 - V^2}{x}$$

$$\therefore \cancel{x^2} \quad \therefore V^2 = \frac{x^2 - L^2}{x^2} \quad \therefore 1 - V^2 = \frac{L^2}{x^2}$$

$$\therefore a = \frac{L^2}{x^3}$$

$$\gamma^3 a = \left(\frac{x}{L}\right)^3 \left(\frac{L^2}{x^3}\right) = \frac{1}{L}$$

$$= \text{const}$$

$$\underline{f} = \frac{d\underline{p}}{dt} = \frac{d}{dt} (\gamma m \underline{v})$$

$$\therefore \gamma m \underline{v} = \underline{f} t + \underline{C}$$

$$\text{At } t=0, \underline{v}=0 \quad \therefore \underline{C} = 0$$

$$\therefore \gamma m \underline{v} = \underline{f} t \quad \underline{f} \text{ and } \underline{v} \text{ are parallel}$$

$$\therefore \text{their magnitudes } \gamma m v = f t$$

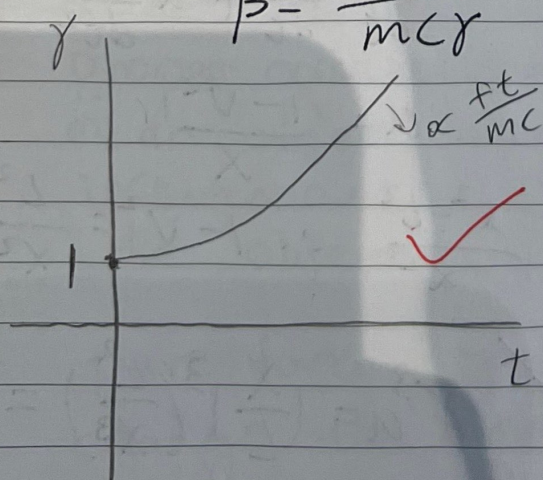
$$\therefore \beta = \frac{v}{c} \quad \therefore \gamma \beta m c = f t$$

$$\therefore \begin{cases} \gamma \beta = \frac{f t}{m c} \\ \gamma^2 (1 - \beta^2) = 1 \end{cases} \quad \therefore \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

$$\therefore \gamma^2 - (\gamma \beta)^2 = 1 \quad \therefore \gamma \beta = \frac{f t}{m c}$$

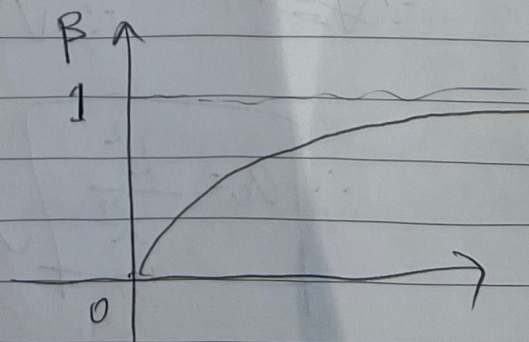
$$\therefore \gamma^2 = 1 + \frac{f^2 t^2}{m^2 c^2}$$

$$\beta = \frac{f t}{m c \gamma}$$



$\therefore$

$$\boxed{\begin{aligned} \gamma &= \sqrt{1 + \left(\frac{f t}{m c}\right)^2} \quad \checkmark \\ \beta &= \frac{f t / m c}{\sqrt{1 + \left(\frac{f t}{m c}\right)^2}} \end{aligned}}$$



- electric force is pure force

$$\begin{aligned} \therefore \frac{dE}{dt} &= \overbrace{\underline{f} \cdot \underline{u}}^{\text{force}} \quad \because E = \gamma mc^2 \\ \text{rate of change of energy} \quad \therefore \frac{d\gamma}{dt} &= \frac{\underline{f} \cdot \underline{u}}{mc^2} \quad \text{velocity} \end{aligned}$$

in this case $\underline{f}$ and $\underline{u}$ are parallel

$$\therefore \underline{f} = e\underline{E} \quad \therefore \frac{d\gamma}{dt} = \frac{eEu}{mc^2}$$

$$d\gamma = \frac{eE}{mc^2} \underbrace{u dt}_{dx} \quad \therefore \frac{d\gamma}{dx} = \frac{eE}{mc^2}$$

$$\therefore \gamma_f = \gamma_i \quad \gamma(x) - \gamma(x=0) = \frac{eEx}{mc^2}$$

- electron ~~se~~ accelerates from rest

$$\therefore \gamma(0) = 1$$

$$\therefore \gamma(L) = 1 + \frac{eEL}{mc^2} = 1 + \frac{(e)(5 \text{ MV/m})(10 \text{ m})}{0.511 \text{ MeV}}$$

$$= 1 + \frac{50}{0.511} = \underline{98.85}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \underline{0.99995}$$

$$\text{If now } E = (1 - 2\%) (5 \text{ MeV/m}) = 4 \text{ MV/m}$$

$$\gamma'(L) = 1 + \frac{40}{0.511} = 79.28$$

$$\beta' = \sqrt{1 - \frac{1}{\gamma^2}} = 0.999992$$

$$\begin{aligned} \beta \text{ changes by } & \frac{|\beta' - \beta|}{\beta} \times 100\% = \\ \text{(reduces)} & \\ & = \underline{\underline{0.003\%}} \end{aligned}$$

$$\text{time takes } ft = \gamma m v = \gamma m \beta c$$

$$f = eE$$

$$\therefore t = \frac{\gamma \beta m c}{eE} \quad \downarrow \beta \sim 1$$

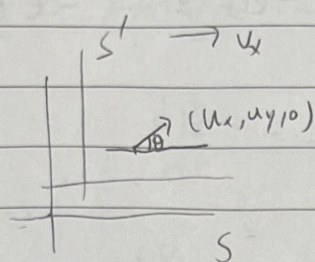
$$= \frac{(98.85) (0.511 \text{ MeV}/c^2) \cdot c}{e (5 \text{ MV/m})}$$

$$= \frac{98.85 \times 0.511}{5} \times \frac{1}{c} \frac{\text{MeV} \cdot \text{m}}{\text{m} \cdot \text{s}}$$

$$= \frac{98.85 \times 0.511}{5} \times \frac{1}{3 \times 10^8} \text{ s}$$

$$= \underline{\underline{3.37 \times 10^{-8} \text{ s}}}$$

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Lorentz transformation

$$t' = \gamma(t - \frac{vx}{c^2})$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$\begin{aligned} u_x' &= \frac{dx'}{dt'} = \frac{dx'/dt}{dt'/dt} = \frac{dx'/dt}{\gamma(1 - \frac{v}{c^2} \frac{dx}{dt})} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \end{aligned}$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy'/dt}{dt'/dt} = \frac{dy'/dt}{\gamma(1 - \frac{v}{c^2} \frac{dx}{dt})} = \frac{u_y}{\gamma(1 - \frac{v u_x}{c^2})}$$

$$\underline{u} = \underline{u}_{\parallel} + \underline{u}_{\perp}$$

$$\Rightarrow u_y = |u| \sin \theta \quad u_x = |u| \cos \theta$$

$$\tan \theta' = \frac{u_y'}{u_x'} = \frac{\frac{u_y}{\gamma(1 - \frac{v u_x}{c^2})}}{\frac{u_x - v}{1 - \frac{u_x v}{c^2}}} = \frac{u_y}{\gamma(u_x - v)}$$

$$\equiv \frac{|u| \sin \theta}{\gamma_v (|u| \cos \theta - v)} = \frac{u \sin \theta}{\gamma_v (u \cos \theta - v)}$$

$$\gamma_v = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$|u| = u$$

— electron in EM field, $\underline{f} = -e(\underline{E} + \underline{v} \times \underline{B})$

$$\therefore \text{pure force} \therefore \frac{d\underline{E}}{dt} = \underline{f} \cdot \underline{u} \quad \therefore E = \gamma m_0 c^2$$

$$\therefore \frac{d\underline{x}}{dt} = \frac{\underline{f} \cdot \underline{u}}{m_0 c^2}$$

$$\underline{f} \cdot \underline{v} = -e(\underline{E} + \underline{v} \times \underline{B}) \cdot \underline{v} = -e\underline{E} \cdot \underline{v} = -eE_z v_z$$

$$\therefore \underline{E} = (0, 0, E_z)^T$$

$$\therefore \frac{d\gamma}{dt} = - \frac{eE_z v_z}{m_0 c^2}$$

$$\therefore d\gamma = - \frac{eE_z}{m_0 c^2} v_z dt = - \frac{eE_z}{m_0 c^2} dz$$

$$\therefore \gamma_f - \gamma_i = - \frac{eE_z}{m_0 c^2} (z_f - z_i)$$

assume at $t=t_0$, $z=0$, $\gamma_i = 1$, $z_f = 1m$

$$\therefore \gamma_f = \gamma_i - \frac{eE_z z_f}{m_0 c^2}$$

$$\underline{p} = \gamma m \underline{v}, \quad \underline{p}_{||} = \gamma m \underline{v}_{||}, \quad \underline{p}_{\perp} = \gamma m \underline{v}_{\perp}$$

($||$ and $\perp$ here are with respect to $\hat{\underline{z}}$, not $\underline{v}$)

$$\underline{v} = \underline{v}_{||} + \underline{v}_{\perp}$$

$$\underline{E}_{\perp} = 0, \quad \underline{E}_{||} = \underline{E}, \quad \underline{B}_{\perp} = 0, \quad \underline{B}_{||} = \underline{B}$$

$$\therefore \underline{f} = -e(\underline{E}_{||} + (\underline{v}_{||} + \underline{v}_{\perp}) \times \underline{B}_{||}) \quad \therefore \underline{v}_{||} \times \underline{B}_{||} = 0$$

$$\therefore \underline{f} = -e(\underline{E}_{||} + \underline{v}_{\perp} \times \underline{B}_{||}) = \frac{d\underline{p}}{dt} = \left(\frac{d\underline{p}_{||}}{dt} + \frac{d\underline{p}_{\perp}}{dt} \right)$$

$$P_{||} = \gamma m \underline{v}_{||} = \gamma m v_z \hat{z}$$

$$\begin{aligned} \frac{dP_{||}}{dt} &= \frac{d}{dt} (\gamma m v_z \hat{z}) \\ &= \frac{d}{dt} (\gamma m v_z) \hat{z} \\ &= \frac{d\gamma}{dt} m v_z \hat{z} + \gamma m \frac{dv_z}{dt} \hat{z} = \frac{d\gamma}{dt} m v_z \hat{z} + \frac{dP_{||}}{dt} \hat{z} \\ &= \frac{dP_{||}}{dt} \hat{z} \end{aligned}$$

$$\therefore \frac{dP_{||}}{dt} = \frac{d}{dt} (\gamma m v_z) \hat{z} \quad \left(\frac{d\hat{z}}{dt} = 0 \right)$$

$$\therefore \frac{dP_{||}}{dt} \text{ always along } \hat{z} \quad (||)$$

$P_{\perp}$ is always $\perp \hat{z}$ at all times

$$\therefore \frac{dP_{\perp}}{dt} \text{ always } \perp \hat{z} \quad (\perp)$$

$$\therefore \underline{E}_{||} \text{ along } \hat{z}, \quad \underline{v}_{\perp} \times \underline{B}_{||} \perp \hat{z}$$

$$\therefore \frac{dP_{||}}{dt} = -e \underline{E}_{||}, \quad \frac{dP_{\perp}}{dt} = -e \underline{v}_{\perp} \times \underline{B}_{||}$$

$$\therefore \underline{P}_{\perp} = \gamma m \underline{v}_{\perp} \quad \therefore \frac{d(P_{\perp}^2)}{dt}$$

$$\begin{aligned} \therefore \frac{d}{dt} (P_{\perp}^2) &= 2 \underline{P}_{\perp} \cdot \frac{d\underline{P}_{\perp}}{dt} = -2 \gamma m e \underbrace{\underline{v}_{\perp} \cdot (\underline{v}_{\perp} \times \underline{B}_{||})}_0 \\ &= 0 \end{aligned}$$

$\therefore |\underline{P}_{\perp}|$ remains constant

$$\text{At } t=t_0, \quad \frac{|P_{||}|}{|P_{\perp}|} = 1 \quad \therefore \frac{\gamma m v_z}{\gamma m v_{\perp}} = 1 = \frac{v_z}{v_{\perp}}$$

$$\therefore v_z = v_{\perp} \text{ at } t=0$$

$$\text{At } t=0 \quad \gamma = 17 = \frac{1}{\sqrt{1-\beta^2}} \quad (\beta = \frac{v}{c})$$

$$\therefore 1-\beta^2 = \frac{1}{17^2} \quad \therefore \beta = \sqrt{1-\frac{1}{17^2}} = 0.998$$

$$\therefore \cancel{V_z(0)} = \cancel{V_z(0)} = 0$$

$$\cancel{V_{t0}} \quad V_{t0} = \cancel{0.998c} = \frac{12\sqrt{3}}{17}c$$

$$V_{t0}^2 = V_z^2(t_0) + V_{\perp}^2(t_0) = 2V_{\perp}^2(t_0) = (0.998)^2 c^2$$

$$\therefore V_{\perp}(t_0) = \frac{0.998c}{\sqrt{2}} = \cancel{0.706c} \frac{12\sqrt{2}}{17}c = \frac{12}{17}c$$

$$|P_{\perp}| = \gamma m_0 V_{\perp} = \gamma(t_0) m_0 V_{\perp}(t_0)$$

$$= (\cancel{17 \times 0.706}) m_0 c = \cancel{12 m_0 c}$$

$$= 17 \times m_0 \times \frac{12}{17} c = 12 m_0 c$$

$$\text{At } z=1m. \quad \gamma = \gamma_f \quad V = V_{\perp} = V_f \quad V_z = 0$$

$$\gamma_f V_f m_0 = \frac{V_f}{\sqrt{1 - V_f^2/c^2}} m_0 = 12 m_0 c = |P_{\perp}| \quad (P_{\parallel} = 0 \text{ at } z=1m)$$

$$\therefore V_f^2 = 12^2 (1 - \frac{V_f^2}{c^2}) c^2$$

$$\therefore V_f^2 = 144 c^2 - 144 V_f^2$$

$$\therefore 145 V_f^2 = 144 c^2 \quad \therefore V_f = \frac{12}{\sqrt{145}} c$$

$$\gamma_f = \frac{1}{\sqrt{1 - \frac{V_f^2}{c^2}}} = \underline{\underline{12.04}}$$

$$\therefore 12.04 - 17 = - \frac{e E_z (1\text{m})}{0.511 \text{ MeV}}$$

$$\begin{aligned} \therefore e E_z (1\text{m}) &= (0.511 \times 10^6) (17 - 12.04) \\ &= 2.535 \times 10^6 \text{ eV} \end{aligned}$$

$$E_z = 2.535 \times 10^6 \text{ V/m}$$

$$\underline{f} = -e(\underline{E}_{\parallel} + \underline{V}_{\perp} \times \underline{B}_{\parallel}) = \gamma m_0 \underline{a} + \frac{\underline{f} \cdot \underline{V}}{c^2} \underline{V}$$

$$\left(\underline{f} = \frac{d}{dt}(\gamma m_0 \underline{V}) = m_0 \left(\gamma \frac{d\underline{V}}{dt} + \underline{V} \frac{d\gamma}{dt} \right) = \gamma m_0 \underline{a} + m_0 \underline{V} \frac{\underline{f} \cdot \underline{V}}{m_0 c^2} \right. \\ \left. = \gamma m_0 \underline{a} + \frac{\underline{f} \cdot \underline{V}}{c^2} \underline{V} \right).$$

$$\therefore -e(\underline{E}_{\parallel} + \underline{V}_{\perp} \times \underline{B}_{\parallel}) = \gamma m_0 (\underline{a}_{\parallel} + \underline{a}_{\perp}) + \frac{e E_z V_z}{c^2} (\underline{V}_{\parallel} + \underline{V}_{\perp})$$

$$\therefore \underline{E}_{\parallel}, \underline{a}_{\parallel}, \underline{V}_{\parallel} \text{ along } \hat{z} \text{ always}$$

$$\underline{V}_{\perp} \times \underline{B}_{\parallel}, \underline{a}_{\perp}, \underline{V}_{\perp} \perp \hat{z} \text{ always}$$

$$\therefore -e \underline{E}_{\parallel} = \gamma m_0 \underline{a}_{\parallel} - \frac{e E_z V_z}{c^2} \underline{V}_{\parallel}$$

$$-e \underline{V}_{\perp} \times \underline{B}_{\parallel} = \gamma m_0 \underline{a}_{\perp} - \frac{e E_z V_z}{c^2} \underline{V}_{\perp}$$

$$\therefore \underline{V}_{\perp} \perp \underline{V}_{\perp} \times \underline{B}_{\parallel} \quad \text{let } \underline{V}_{\perp} \times \underline{B}_{\parallel} \text{ along direction } \hat{r}$$

$$\hat{r} = \frac{\underline{V}_{\perp} \times \hat{z}}{|\underline{V}_{\perp} \times \hat{z}|}$$

$$\text{and } \underline{V}_{\perp} \text{ along } \hat{\phi} = \frac{e \underline{V}_{\perp} \times \hat{z}}{|\underline{V}_{\perp}|}$$

then $\underline{V}_\perp \times \underline{B} = \underline{V}_\perp B \underline{\hat{r}} \quad \underline{V}_\perp = V_\perp \underline{\hat{v}}$

$\underline{\hat{r}} \perp \underline{\hat{v}} \quad , \text{ let } \underline{a}_\perp = a_r \underline{\hat{r}} + a_v \underline{\hat{v}}$

$\therefore -e V_\perp B \underline{\hat{r}} = \gamma m (a_r \underline{\hat{r}} + a_v \underline{\hat{v}}) - \frac{e E_z v_z}{c^2} V_\perp \underline{\hat{v}}$

$\therefore -e V_\perp B \underline{\hat{r}} = \gamma m a_r \underline{\hat{r}}$

$\gamma m a_v \underline{\hat{v}} = \frac{e E_z v_z V_\perp}{c^2} \underline{\hat{v}}$

$\therefore e V_\perp B = \gamma m |a_r|$

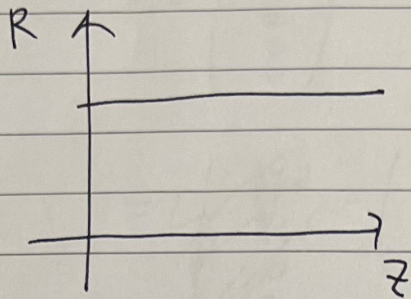
$|a_r|$ is the magnitude of acceleration perpendicular to both velocity and $\underline{B}$

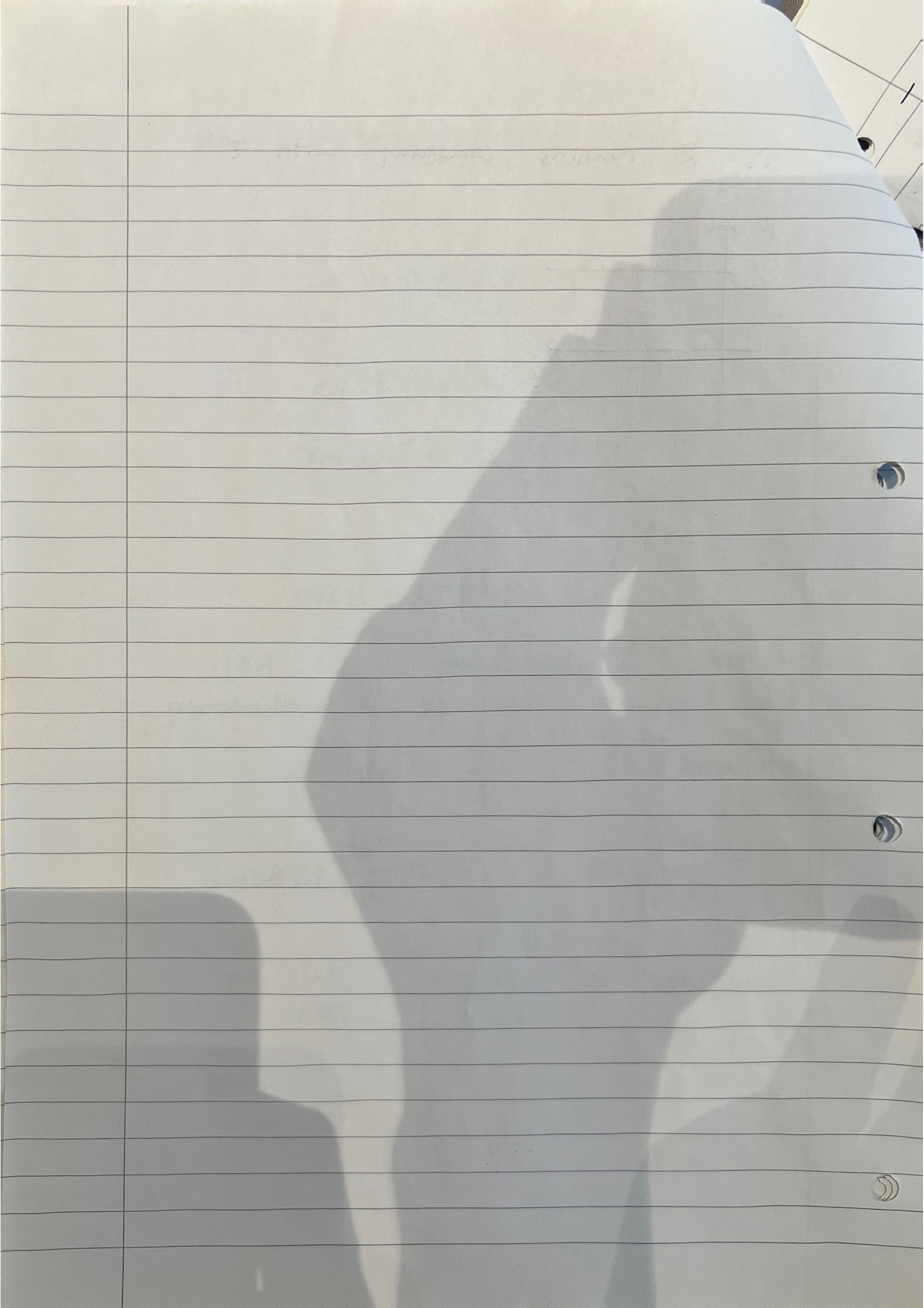
$\therefore |a_r| = \frac{V_\perp^2}{R} \quad (R = \text{Larmor radius})$

$e V_\perp B = \frac{\gamma V_\perp^2}{R} m_0 \quad \therefore R = \frac{\gamma m_0 V_\perp}{e B}$

$R = \frac{|P_\perp|}{e B} \quad \therefore |P_\perp| \text{ is constant}$

$\therefore R$ remains constant with z





Appendix

"Brute-force" method of Q2 part ii (2016 B2):

$$-\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2} \underline{v} \cdot \underline{\dot{v}} \quad (\underline{\dot{v}} = \frac{d\underline{v}}{dt})$$

$$\frac{d\underline{p}}{dt} = -e(\underline{E} + \underline{v} \times \underline{B})$$

$$\underline{v} = (v_x, v_y, v_z)^T \quad \underline{B} = (0, 0, B)^T, \quad \underline{E} = (0, 0, E_z)^T$$

$$\underline{v} \times \underline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = (Bv_y, -Bv_x, 0)^T$$

$$\therefore \frac{d\underline{p}}{dt} = -e(Bv_y, -Bv_x, E_z)^T$$

$$\therefore \underline{p} = \gamma m_0 \underline{v} = \gamma m_0 (v_x, v_y, v_z)^T$$

$$\text{where } \gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} = (1 - \frac{v_x^2 + v_y^2 + v_z^2}{c^2})^{-\frac{1}{2}}$$

$$\therefore \gamma \frac{dv_x}{dt} = -\omega v_y$$

$$\gamma \frac{dv_y}{dt} = \omega v_x$$

$$\gamma \frac{dv_z}{dt} = -A$$

constants

$$\omega \equiv \frac{eB}{m_0}$$

$$A \equiv \frac{eE_z}{m_0}$$

$$\omega, A > 0$$

$$\frac{d}{dt}(\gamma v_x) = -\omega v_y \quad (1)$$

$$\frac{d}{dt}(\gamma v_y) = \omega v_x \quad (2)$$

$$\frac{d}{dt}(\gamma v_z) = -A \quad (3)$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_x^2 + v_y^2 + v_z^2}{c^2}}} \quad (4)$$

$$\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2} \underline{v} \cdot \underline{\dot{v}}$$

$$\textcircled{3} \quad \gamma \frac{dV_z}{dt} + V_z \frac{d\gamma}{dt} = -A$$

$$\frac{d\gamma}{dt} = \frac{\gamma^2}{c^2} \left[\gamma V_x \frac{dV_x}{dt} + \gamma V_y \frac{dV_y}{dt} + \gamma V_z \frac{dV_z}{dt} \right]$$

~~$$\frac{\gamma^2}{c^2} \textcircled{1} \quad \gamma \frac{dV_x}{dt} + V_x \frac{d\gamma}{dt} = -\omega V_y$$~~

~~$$\textcircled{2} \quad \gamma \frac{dV_y}{dt} + V_y \frac{d\gamma}{dt} = \omega V_x$$~~

$$\therefore \frac{d\gamma}{dt} = \frac{\gamma^2}{c^2} \left[-\cancel{\omega V_y V_x} - V_x^2 \frac{d\gamma}{dt} + \cancel{\omega V_x V_y} - V_y^2 \frac{d\gamma}{dt} - A V_z - V_z^2 \frac{d\gamma}{dt} \right]$$

$$= \frac{\gamma^2}{c^2} \left[-(V^2) \frac{d\gamma}{dt} - A V_z \right]$$

$$\therefore \left(1 + \frac{\gamma^2 V^2}{c^2} \right) \frac{d\gamma}{dt} = -A V_z \frac{\gamma^2}{c^2}$$

$$\left(1 - \frac{V^2}{c^2} \right)^{\frac{1}{2}} = \frac{1}{\gamma^2} \quad \therefore \quad \frac{V^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

$$\therefore \left(1 + \left(1 - \frac{1}{\gamma^2} \right) \gamma^2 \right) \frac{d\gamma}{dt} = -A V_z \frac{\gamma^2}{c^2}$$

$$\therefore \gamma^2 \frac{d\gamma}{dt} = -\frac{A}{c^2} \gamma^2 V_z$$

$$\therefore \boxed{\frac{d\gamma}{dt} = -\frac{A}{c^2} V_z} \equiv -K V_z \quad \left(K \equiv \frac{A}{c^2} \equiv \frac{e E_z}{m_0 c^2} \right)$$

$$\therefore d\gamma = -K V_z dt = -K dz$$

$$\therefore \Delta\gamma = -K \Delta z$$

$$\therefore \underbrace{\gamma(z) - \gamma(z=0)}_{17} = -K(z-0)$$

$$\therefore \gamma(z) = 17 - \frac{e E_z z}{m_0 c^2}$$

$$\boxed{\gamma(1m) = 17 - \frac{e E_z (1m)}{m_0 c^2}}$$

focus on ①, ②

$$V_x \textcircled{1} + V_y \textcircled{2} \Rightarrow \gamma \left(V_x \frac{dV_x}{dt} + V_y \frac{dV_y}{dt} \right) + (V_x^2 + V_y^2) \frac{d\gamma}{dt} = 0$$

$$\text{Define } V_{\perp} = \sqrt{V_x^2 + V_y^2} \quad \therefore \frac{d}{dt}(V_{\perp}^2) = 2V_x \frac{dV_x}{dt} + 2V_y \frac{dV_y}{dt}$$

$$\therefore \frac{1}{2} \gamma \frac{d}{dt}(V_{\perp}^2) + V_{\perp}^2 \frac{d\gamma}{dt} = 0$$

$$\therefore \gamma V_{\perp} \frac{dV_{\perp}}{dt} + V_{\perp}^2 \frac{d\gamma}{dt} = 0$$

$$\therefore \gamma \frac{dV_{\perp}}{dt} + V_{\perp} \frac{d\gamma}{dt} = 0$$

$$\therefore \boxed{\frac{d}{dt}(\gamma V_{\perp}) = 0} \Rightarrow \boxed{V_{\perp} \gamma \equiv b}$$

$$b = \text{const}$$

$$\therefore \frac{1}{\sqrt{1 - \frac{V_1^2 + V_2^2}{c^2}}} V_1 = b$$

$$\therefore V_1^2 = b^2 \left(1 - \frac{V_1^2 + V_2^2}{c^2} \right)$$

$$\therefore V_1^2 = b^2 - \frac{b^2}{c^2} V_1^2 - \frac{b^2}{c^2} V_2^2$$

$$\therefore \left(1 + \frac{b^2}{c^2} \right) V_1^2 + \left(\frac{b^2}{c^2} \right) V_2^2 = b^2$$

$$\times \left(\frac{c}{b} \right)^2 \Rightarrow \boxed{\left(1 + \frac{c^2}{b^2} \right) V_1^2 + V_2^2 = c^2}$$

$$\text{define } m^2 = \frac{c^2}{b^2} \quad \boxed{S^2 = 1 + \frac{c^2}{b^2}} \text{ then}$$

$$\boxed{S^2 V_1^2 + V_2^2 = c^2}$$

$$\therefore \frac{d}{dt} (\gamma V_2) = -A \quad (3)$$

$$\therefore \gamma V_2 = -At + B$$

$$\text{If at } t=t_0, V_2=0, \text{ then } B = At_0$$

$$\therefore \gamma V_2 = A(t_0 - t)$$

$$\boxed{\alpha = \frac{A}{b}}$$

$$\therefore \gamma V_2 = A(t_0 - t)$$

$$\gamma V_1 = b$$

$$\therefore \frac{V_1}{V_2} = \frac{b}{A(t_0 - t)} = \frac{1}{\alpha(t_0 - t)}$$

$$\therefore V_z = \alpha(t_0 - t) V_L, \quad V_L = \frac{V_z}{\alpha(t_0 - t)}$$

$$\therefore \cancel{[S^2 \alpha^2 (t_0 - t)^2 + 1]} V_L^2 = C^2$$

$$\therefore \cancel{V_L} = \frac{C}{\cancel{\sqrt{1 + \alpha^2 S^2 (t_0 - t)^2}}}$$

$$\therefore S^2 V_L^2 + \alpha^2 (t_0 - t)^2 V_L^2 = C^2$$

$$\therefore V_L = \frac{C}{\sqrt{S^2 + \alpha^2 (t_0 - t)^2}} = \frac{C/S}{\sqrt{1 + (\frac{\alpha}{S})^2 (t_0 - t)^2}}$$

$$\text{define } \delta \equiv \frac{\alpha}{S}, \quad g \equiv \frac{C}{S}$$

$$V_L = \frac{g}{\sqrt{1 + \delta^2 (t_0 - t)^2}}$$

$$(S = \frac{\alpha}{\delta} > 1)$$

$$(\alpha > \delta)$$

$$(C = gS = g\frac{\alpha}{\delta})$$

$$S^2 \frac{V_z^2}{(\alpha(t_0 - t))^2} + V_z^2 = C^2$$

$$\therefore \left(1 + \frac{1}{\delta^2 (t_0 - t)^2}\right) V_z^2 = C^2$$

$$V_z = \frac{C}{\sqrt{1 + \frac{1}{\delta^2 (t_0 - t)^2}}} = \frac{C \cdot \delta (t_0 - t)}{\sqrt{1 + \delta^2 (t_0 - t)^2}}$$

$$C\delta = \frac{C\alpha}{S} = \alpha g$$

$$= \frac{\alpha(t_0 - t)g}{\sqrt{1 + \delta^2 (t_0 - t)^2}}$$

Now for $V_x(t)$ & $V_y(t)$

$$\therefore V_x^2(t) + V_y^2(t) = V_L^2(t)$$

$$\therefore \text{let } \begin{aligned} V_x(t) &= V_L(t) \cos(f(t)) \\ V_y(t) &= V_L(t) \sin(f(t)) \end{aligned}$$

$$\begin{aligned} V_x(t) &= V_L(t) \cos(f(t)) \\ V_y(t) &= V_L(t) \sin(f(t)) \end{aligned} \quad (\gamma V_L = b)$$

$$\frac{d}{dt}(\gamma V_x) = -\omega V_y$$

$$\therefore -\gamma V_L \sin(f(t)) \frac{df}{dt} = -\omega V_L \sin(f(t))$$

$$\therefore \frac{df}{dt} = \frac{\omega}{\gamma} = \omega \sqrt{1 - \frac{v^2}{c^2}} = \omega \sqrt{1 - \frac{v_x^2 + v_y^2}{c^2}}$$

similarly $\frac{d}{dt}(\gamma V_y) = \omega V_x$

$$\gamma V_L \cos(f(t)) \frac{df}{dt} = \omega V_L \cos(f(t))$$

$$\frac{df}{dt} = \frac{\omega}{\gamma}$$

$$V_L^2 + V_z^2 = \underbrace{S^2 V_L^2 + V_z^2}_{c^2} - (S^2 - 1) V_L^2$$

$$= c^2 - (S^2 - 1) V_L^2$$

$$= c^2 - k^2 V_L^2 = c^2 (1 - k^2 V_L^2)$$

$$c^2 k^2 \equiv S^2 - 1$$

$$k^2 \equiv S^2 - 1$$

$$\therefore 1 - \frac{v_L^2 + v_z^2}{c^2} = 1 - (1 - k^2 v_L^2) = (k v_L)^2$$

$$\therefore \frac{df}{dt} = (\omega k) v_L$$

$$\therefore \cancel{f(t) = C + \int}$$

$$f(t) = C + \omega k \int v_L(t) dt$$

$$\int v_L(t) dt = \int \frac{q}{\sqrt{1 + \delta^2 (t_0 - t)^2}} dt$$

$$\text{let } \delta(t_0 - t) = \tan \theta$$

$$-\delta dt = \sec^2 \theta d\theta$$

$$dt = -\frac{1}{\delta} \sec^2 \theta d\theta$$

$$= -\frac{q}{\delta} \int \frac{\sec^2 \theta d\theta}{\sqrt{1 + \tan^2 \theta}}$$

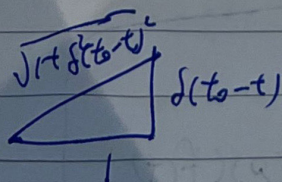
$$= -\frac{q}{\delta} \int \frac{\sec^2 \theta d\theta}{\sec \theta} = -\frac{q}{\delta} \int \sec \theta d\theta$$

~~tan~~

$$= -\frac{q}{\delta} \int \frac{\sec \theta (\sec \theta + \tan \theta) d\theta}{\sec \theta + \tan \theta}$$

$$= -\frac{q}{\delta} \int \frac{d(\sec \theta + \tan \theta)}{\sec \theta + \tan \theta} = -\frac{q}{\delta} \ln |\sec \theta + \tan \theta|$$

$$= -\frac{q}{\delta} \ln \left(\sqrt{1 + \delta^2 (t_0 - t)^2} + \delta(t_0 - t) \right)$$



$$f(0) = 0 \Rightarrow$$

$$\cancel{f(t) =}$$

$$f(t) = \omega_k \left[\frac{q}{\delta} \right] \left[\ln \frac{\sqrt{1 + \delta^2 t_0^2} + \delta t_0}{\sqrt{1 + \delta^2 (t_0 + t)^2} + \delta (t_0 + t)} \right]$$

$$f(0) = 0 \quad \therefore \quad V_x(0) = V_{\perp}(0) \quad V_y(0) = 0 \quad (\text{WLG})$$

$$V_{\perp} = \frac{1}{\omega_k} \frac{df}{dt}$$

$$\therefore V_x(t) = V_{\perp}(t) \cos(f(t)) \quad V_y = V_{\perp}(t) \sin(f(t))$$

$$= \frac{1}{\omega_k} \frac{df}{dt} \cos(f)$$

$$= \frac{1}{\omega_k} \frac{df}{dt} \sin(f)$$

$$X(t) = \int V_x(t) dt = \frac{1}{\omega_k} \int \frac{df}{dt} \cos(f) dt$$

$$= \frac{1}{\omega_k} \int \cos(f) df = \frac{1}{\omega_k} \sin(f) + C'$$

$$X(0) = X_0 \quad \therefore \quad C' = X_0$$

$$\therefore X = \frac{1}{\omega_k} \sin(f(t)) + X_0$$

$$Y(t) = \int V_y(t) dt = \frac{1}{\omega_k} \int \sin(f) df = -\frac{1}{\omega_k} \cos(f) + C''$$

$$Y(0) = Y_0 \quad \therefore \quad -\frac{1}{\omega_k} + C'' = Y_0 \quad C'' = Y_0 + \frac{1}{\omega_k}$$

$$\therefore Y = \left(Y_0 + \frac{1}{\omega_k} \right) - \frac{1}{\omega_k} \cos(f)$$

$$\therefore \quad \boxed{\begin{aligned} (X - X_0) &= \frac{1}{\omega_k} \sin(f(t)) \\ (Y - (Y_0 + \frac{1}{\omega_k})) &= -\frac{1}{\omega_k} \cos(f(t)) \end{aligned}}$$

$$\therefore (x-x_0)^2 + (y-(y_0 + \frac{1}{\omega k}))^2 = \frac{1}{\omega^2 k^2} = R^2$$

$$R = \frac{1}{\omega k}$$

$$\frac{1}{\gamma} = k v_L \quad \therefore \frac{1}{k} = \gamma v_L = \text{const}$$

$$R = \frac{\gamma v_L}{\omega} = \frac{\gamma v_L}{\frac{eB}{m}} = \boxed{\frac{\gamma m v_L}{eB}}$$

Circular motion $\Rightarrow$ unchanged Larmor Radius

$$\gamma = \frac{1}{k v_L} \quad \frac{d\gamma}{dt} = -\frac{1}{k v_L^2} \frac{d v_L}{dt} = -k v_z$$

$$\begin{aligned} \frac{d(\gamma v_L)}{dt} &= \frac{d\gamma}{dt} v_L + \gamma \frac{d v_L}{dt} \\ &= \gamma \frac{d v_L}{dt} - k v_L v_z \end{aligned}$$

$$z(t) = \int v_z dt = \alpha_L \int \frac{(t_0 - t)}{\sqrt{1 + \delta^2 (t_0 - t)^2}} dt = C \int \frac{\delta(t_0 - t)}{\sqrt{1 + \delta^2 (t_0 - t)^2}} dt$$

$$\begin{aligned} \text{let } u &= \delta(t_0 - t) \\ du &= -\delta dt \\ \therefore dt &= -\frac{1}{\delta} du \end{aligned} \quad = -\frac{C}{\delta} \int \frac{u du}{\sqrt{1+u^2}} = -\frac{C}{2\delta} \int \frac{dJ}{\sqrt{J}}$$

$$\begin{aligned} \text{let } J &= 1+u^2 \\ dJ &= 2u du \\ u du &= \frac{1}{2} dJ \end{aligned} \quad = -\frac{C}{\delta} \sqrt{J} = -\frac{C}{\delta} \sqrt{1 + \delta^2 (t_0 - t)^2} + C^{(2)}$$

$$C^{(2)} = \frac{C}{\delta} \sqrt{1 + \delta^2 t_0^2} \quad \text{for } z(0) = 0$$

$$\therefore \boxed{z(t) = \frac{C}{\delta} \left(\sqrt{1 + \delta^2 t_0^2} - \sqrt{1 + \delta^2 (t_0 - t)^2} \right)}$$

$$\therefore (x-x_0) = \left(\frac{1}{k} - \frac{1}{k_0} \right) = \frac{1}{k} - \frac{1}{k_0} = B_0$$

$$R = \frac{1}{k} = \frac{1}{k_0} \therefore \frac{1}{k} = \frac{1}{k_0} = \text{const}$$

$$R = \frac{\frac{1}{k}}{\frac{1}{k_0}} = \frac{\frac{1}{k}}{\frac{1}{k_0}} = \frac{k_0}{k}$$

Explaination: $\Rightarrow$ unchanged parameter

$$R = \frac{1}{k} = \frac{1}{k_0} \Rightarrow \frac{1}{k} = \frac{1}{k_0} = -k_0$$

$$\frac{1}{k} = \frac{1}{k_0} = \frac{1}{k_0} = \frac{1}{k_0}$$

$$= \frac{1}{k} = \frac{1}{k_0} = \frac{1}{k_0}$$

$$\left. \begin{aligned} \frac{1}{k} &= \frac{1}{k_0} \\ \frac{1}{k} &= \frac{1}{k_0} \end{aligned} \right\} \Rightarrow \frac{1}{k} = \frac{1}{k_0} = \frac{1}{k_0}$$

$$\left. \begin{aligned} \frac{1}{k} &= \frac{1}{k_0} \\ \frac{1}{k} &= \frac{1}{k_0} \end{aligned} \right\} \Rightarrow \frac{1}{k} = \frac{1}{k_0} = \frac{1}{k_0}$$

$$\left. \begin{aligned} \frac{1}{k} &= \frac{1}{k_0} \\ \frac{1}{k} &= \frac{1}{k_0} \end{aligned} \right\} \Rightarrow \frac{1}{k} = \frac{1}{k_0} = \frac{1}{k_0}$$

$$\left. \begin{aligned} \frac{1}{k} &= \frac{1}{k_0} \\ \frac{1}{k} &= \frac{1}{k_0} \end{aligned} \right\} \Rightarrow \frac{1}{k} = \frac{1}{k_0} = \frac{1}{k_0}$$

— two photons

$$\hbar = c = 1$$

$$P_1 = \begin{pmatrix} E_1 \\ E_1 \\ 0 \\ 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} E_2 \\ E_2 \cos \theta \\ E_2 \sin \theta \\ 0 \end{pmatrix}$$

$$= 2 \begin{pmatrix} \omega \\ \omega \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \omega \\ \omega \cos \theta \\ \omega \sin \theta \\ 0 \end{pmatrix}$$

$$P_{\text{tot}} = P_1 + P_2$$

$$E_{\text{cm}}^2 = -P_{\text{tot}} \cdot P_{\text{tot}}$$

$$= -P_1^2 - P_2^2 - 2P_1 \cdot P_2$$

$$= -0 - 0 - 2P_1 \cdot P_2$$

($P_1^2 = P_2^2 = 0$
for photons)

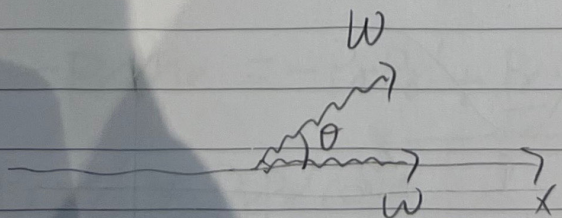
$$= -2[-\omega^2 + \omega^2 \cos \theta]$$

$$= 2\omega^2(1 - \cos \theta)$$

$$\therefore \cancel{E_{\text{cm}} = \sqrt{2}\omega}$$

$$\underline{E_{\text{cm}} = \hbar \omega \sqrt{2(1 - \cos \theta)}}$$

$$\theta \in [0, 2\pi]$$



velocity of CM frame is

$$\underline{V_{cm}} = \frac{\underline{P_{tot}}}{E_{tot}}$$

$$P_1 = \begin{pmatrix} \omega \\ \omega \\ 0 \\ 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} \omega \\ \omega \cos \theta \\ \omega \sin \theta \\ 0 \end{pmatrix}$$

$$\therefore \underline{P_{tot}} = \begin{pmatrix} E_{tot} \\ \underline{P_{tot}} \end{pmatrix}$$

$$\therefore \underline{P_{tot}} = \begin{pmatrix} \omega(1+\cos\theta) \\ \omega \sin\theta \\ 0 \end{pmatrix}$$

$$E_{tot} = 2\omega$$

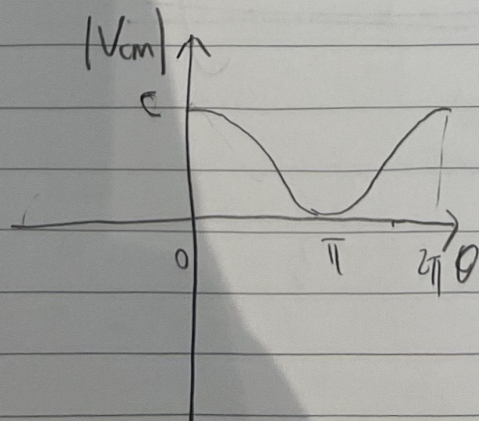
~~$$\underline{V_{cm}} =$$~~

$$\therefore \underline{V_{cm}} = \frac{1}{2\omega} \begin{pmatrix} \omega(1+\cos\theta) \\ \omega \sin\theta \\ 0 \end{pmatrix} C$$

$$\underline{V_{cm}} = \frac{C}{2} \begin{pmatrix} 1+\cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

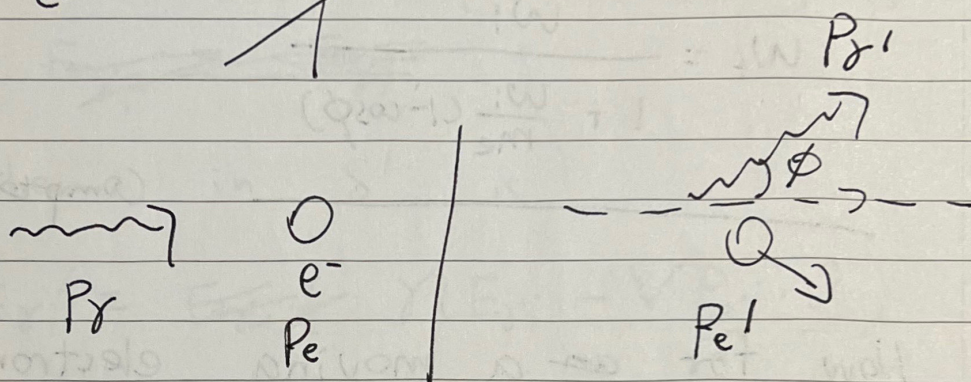
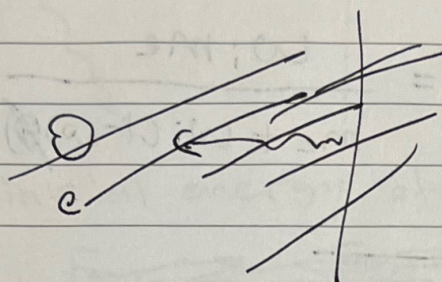
$$|V_{cm}| = \frac{C}{2} \sqrt{(1+\cos\theta)^2 + \sin^2\theta} = \frac{1}{2} C \sqrt{2+2\cos\theta}$$

$$= \frac{1}{2} C \sqrt{2(1+\cos\theta)}$$



- inverse Compton scattering

We start by deriving the Compton effect formula in electron rest frame



4-vector

$$p_r + p_e = p_r' + p_e'$$

$$p_r = \begin{pmatrix} p_r \\ p_r \end{pmatrix} \quad p_e = \begin{pmatrix} m_e \\ 0 \end{pmatrix} \quad p_r' = \begin{pmatrix} p_r' \\ p_r' \end{pmatrix} \quad p_e' = \begin{pmatrix} E_e' \\ p_e' \end{pmatrix}$$

$$p_e' \cdot p_e' = (p_r + p_e - p_r')^2$$

$$-m_e^2 = -m_e^2 + 2p_r \cdot p_e - 2p_r \cdot p_r' - 2p_e \cdot p_r'$$

$$\therefore p_r \cdot p_e = p_r \cdot p_r' + p_e \cdot p_r'$$

$$\therefore -p_r m_e = -p_r p_r' + \underline{p_r} \cdot \underline{p_r'} - m_e p_r'$$

$$\underline{p_r} \cdot \underline{p_r'} = p_r p_r' \cos \phi$$

$$\therefore p_r m_e = p_r p_r' (1 - \cos \phi) + m_e p_r'$$

$$\therefore p_x' = \frac{p_x m_e}{p_x(1 - \cos\theta) + m_e} \quad c = h = 1$$

$$\therefore p_x' = \omega_f', \quad p_x = \omega_i$$

$$\therefore \omega_f = \frac{\omega_i m_e}{m_e + \omega_i(1 - \cos\theta)}$$

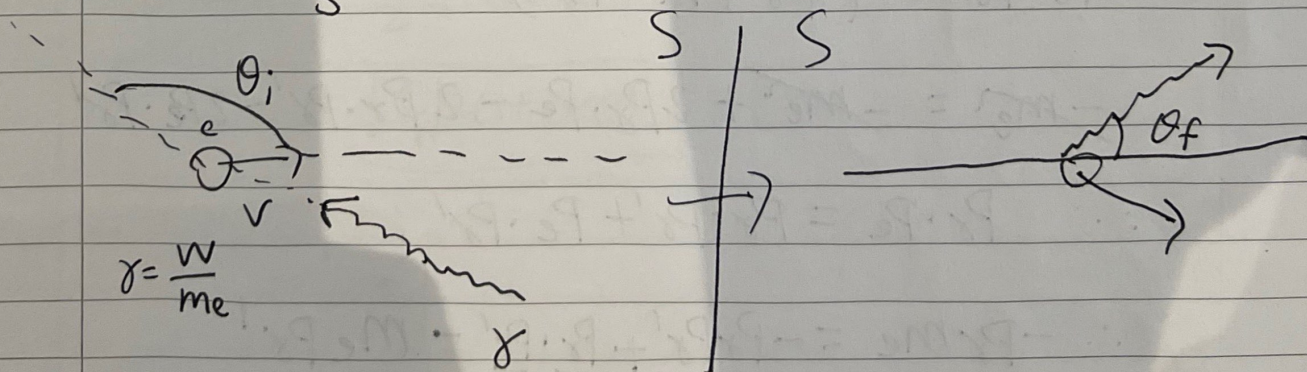
$$\omega_f = \frac{\omega_i}{1 + \frac{\omega_i}{m_e}(1 - \cos\theta)}$$

Compton Scattering

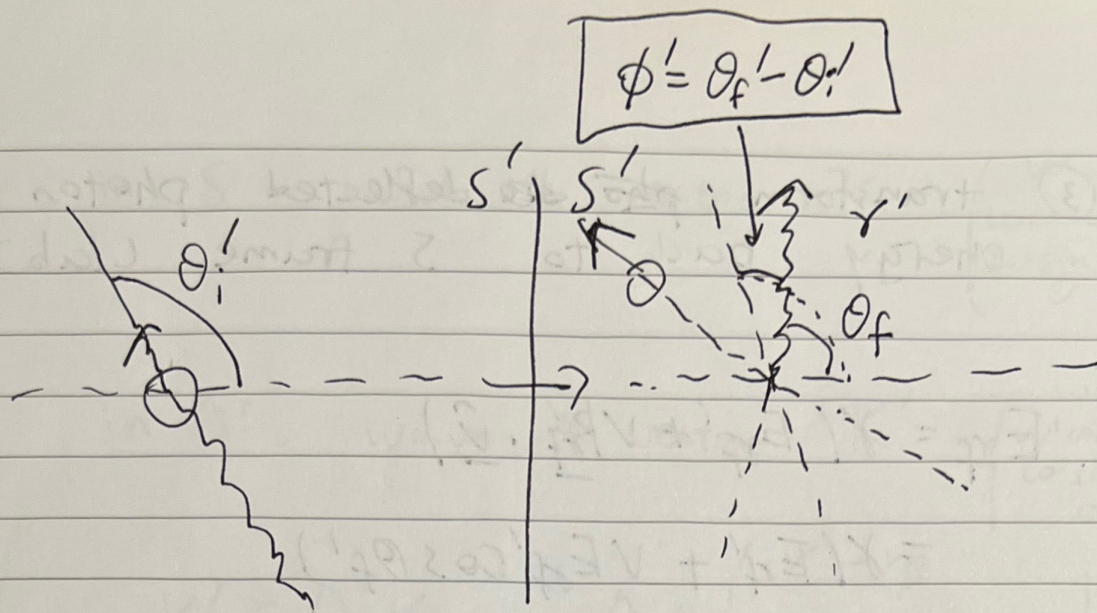
Now for ~~an~~ a moving electron, the photon can gain energy.

We do calculation by

- ① transform photon energy from lab frame S to electron rest frame S'



θ_i, θ_f are the incident and scattered angle, with respect to x -direction of the ~~p~~ photon in S .



① initial energy of photon in S is $E_{\gamma i}$

② ~~$E_{\gamma f}$~~ ~~$E_{\gamma i}$~~

then in S' is

$$E_{\gamma i}' = \cancel{E_{\gamma i}} \gamma (E_{\gamma i} - \underline{V} \underline{P_{\gamma i}} \cdot \underline{\hat{V}})$$

$$= \gamma (E_{\gamma i} - V E_{\gamma i} \cos \theta_i)$$

$$= E_{\gamma i} \gamma (1 - V \cos \theta_i)$$

② in S' , use Compton effect

$$E_{\gamma f}' = \frac{E_{\gamma i}'}{1 + \frac{E_{\gamma i}'}{m_e} (1 - \cos(\phi'))}$$

$$= \frac{E_{\gamma i}'}{1 + \frac{E_{\gamma i}'}{m_e} (1 - \cos(\theta_f' - \theta_i'))}$$

③ transform ~~pho~~ ~~to~~ deflected photon energy back to S frame (lab)

$$E_{xf} = \gamma (E_{xf}' + V \underline{p}_{xf}' \cdot \underline{\hat{v}})$$

$$= \gamma (E_{xf}' + V E_{xf}' \cos \theta_f')$$

$$= \gamma E_{xf}' (1 + V \cos \theta_f')$$

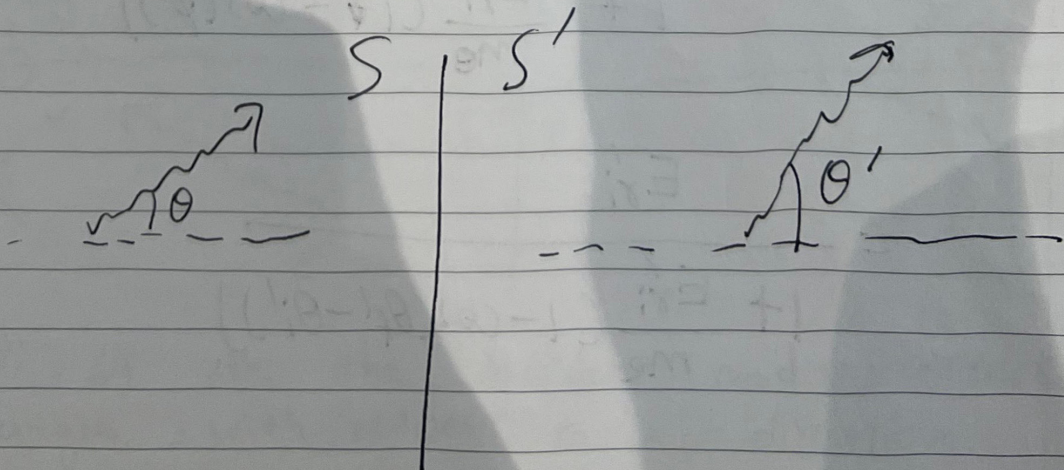
$$= \underline{\gamma E_{xi}' (1 + V \cos \theta_f')}$$

$$1 + \frac{E_{xi}'}{m_e} (1 - \cos(\theta_f' - \theta_i'))$$

$$= \boxed{\frac{E_{xi} \gamma^2 (1 + V \cos \theta_f') (1 - V \cos \theta_i')}{1 + \frac{E_{xi} \gamma}{\gamma m_e} (1 - V \cos \theta_i') (1 - \cos(\theta_f' - \theta_i'))}}$$

Now, consider the relation between

$\theta_{i,f}'$ and $\theta_{i,f}$



in S , wave 4 vector $K = \begin{pmatrix} \omega \\ \omega \cos \theta \\ \omega \sin \theta \\ 0 \end{pmatrix}$

in S' , wave 4 vector $K' = \begin{pmatrix} \omega' \\ \omega' \cos \theta' \\ \omega' \sin \theta' \\ 0 \end{pmatrix}$

$$K' = \Lambda K \quad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \begin{aligned} \omega' &= \gamma\omega - \gamma\beta\omega \cos \theta \\ \omega' \cos \theta' &= -\gamma\beta\omega + \gamma\omega \cos \theta \end{aligned}$$

$$\therefore (\gamma\omega - \gamma\beta\omega \cos \theta) \cos \theta' = -\gamma\beta\omega + \gamma\omega \cos \theta$$

$$\therefore (1 - \beta \cos \theta) \cos \theta' = \cos \theta' - \beta$$

$$C=1 \rightarrow \beta = V$$

$$\therefore \cos \theta'_{if} = \frac{\cos \theta_{if} - V}{1 - V \cos \theta_{if}}$$

$-E_{\gamma i} =$ time component of 4-vector K
 $= \underline{K^0}$

- electron energy $= W \quad \therefore \gamma = \underline{\frac{W}{m_e}}$

→ maximum energy of photon after scattering is the maximum value of

E_{eff} for all possible θ_i, θ_f

This is complicated for general θ

W and V and K^0 ^{I wrote a matlab code for this, result is interesting.}

→ but if we use the numbers in the problem

$$W = 2 \text{ GeV}, \quad m_e = 0.511 \text{ MeV}$$

$$\therefore \gamma = \frac{2 \times 10^9}{0.511 \times 10^6} = 3914 \gg 1$$

$$\therefore V \sim C \sim 1$$

$$K^0 = E_{\gamma_i} = E_{\text{photon}} = \frac{hc}{\lambda} = \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{(1 \times 10^{-2} \text{ m})}$$

$$= 1.99 \times 10^{-23} \text{ J} = 1.243 \times 10^{-4} \text{ eV}$$

$$\therefore K^0 \gamma \approx 0.5 \text{ eV} \ll 0.511 \text{ MeV} = m_e$$

$$\therefore K^0 \gamma \ll m_e \quad \therefore \frac{K^0 \gamma}{m_e} \ll 1$$

→ the denominator of E_{eff} can be treated as 1 and $V \sim 1$

$$\rightarrow E_{sf} \approx K^0 \gamma^2 (1 + v \cos \theta_f') (1 - v \cos \theta_i)$$

this clearly has maximum when

$$\theta_f' = 0^\circ \quad \text{and} \quad \theta_i = 180^\circ$$

$$\Rightarrow \text{corresponds to } (\theta_i, \theta_f) = \text{~~0^\circ, 180^\circ~~} (180^\circ, 0^\circ)$$

$$(\theta_i', \theta_f') = (180^\circ, 0^\circ)$$

$$\therefore \phi' = \phi = \text{~~0^\circ~~} - 180^\circ$$

→ This means the maximum scattered photon energy is obtained when the photon is bounced straight back along the original path.

In this case

$$E_{sf, \max} \approx K^0 (\gamma^2) (1+1) (1+1)$$

$$= \underline{\underline{4\gamma^2 K^0}} = \underline{\underline{4\left(\frac{W}{m_e c^2}\right)^2 K^0}}$$

- with numbers

$$E_{sf, \max} = 4(3914)^2 (1.243 \times 10^{-4} \text{ eV})$$

$$= 7616 \text{ eV}$$

$$= \boxed{7.62 \times 10^3 \text{ eV}} = 1.63 \times 10^{-10} \text{ m} = 0.16 \text{ nm}$$

- For fixed $\theta_i = 180^\circ$ (head-on collision)

$$E_{xf} = \frac{\gamma^2 E_{xi} (1 + v \cos \theta_f') (1 + v)}{1 + \frac{\gamma E_{xi}}{m_e} (1 + v) (1 + \cos \theta_f')} \quad \cos \theta_f' = \frac{\cos \theta_f - v}{1 - v \cos \theta_f}$$

$$\therefore 1 + v \cos \theta_f' = \frac{1 - v \cos \theta_f + v \cos \theta_f - v^2}{1 - v \cos \theta_f} = \frac{1 - v^2}{1 - v \cos \theta_f}$$

$$1 + \cos \theta_f' = \frac{1 - v \cos \theta_f + \cos \theta_f - v}{1 - v \cos \theta_f} = \frac{(1 - v)(1 + \cos \theta_f)}{1 - v \cos \theta_f}$$

$$\therefore E_{xf} = \frac{\gamma^2 E_{xi} (1 - v^2) (1 + v)}{(1 - v \cos \theta_f) \left[1 + \frac{\gamma E_{xi}}{m_e} (1 + v) \frac{(1 - v)(1 + \cos \theta_f)}{1 - v \cos \theta_f} \right]}$$

$$= \frac{\gamma^2 E_{xi} (1 - v^2) (1 + v)}{1 - v \cos \theta_f + \frac{\gamma E_{xi}}{m_e} (1 + v)(1 - v)(1 + \cos \theta_f)} \quad \because \gamma^2 (1 - v^2) = 1$$

$$= \frac{E_{xi} (1 + v)}{1 - v \cos \theta_f + \frac{\gamma E_{xi}}{m_e} (1 + v)(1 - v)(1 + \cos \theta_f)} \quad \times \gamma (1 + v) m_e$$

$$= \frac{E_{xi} \gamma m_e (1 + v)^2}{(1 - v \cos \theta_f) \gamma m_e (1 + v) + \underbrace{E_{xi} \gamma^2 (1 - v^2)}_1 (1 + v) (1 + \cos \theta_f)}$$

$$= \frac{\gamma m_e E_{xi} (1 + v)}{\gamma m_e (1 - v \cos \theta_f) + E_{xi} (1 + \cos \theta_f)}$$

$$= \frac{\gamma m_e E_{xi} (1 + v)}{(E_{xi} + \gamma m_e) + (E_{xi} - \gamma m_e) \cos \theta_f}$$

If $E_{xi} < \gamma m_e$, then $\cos \theta_f = 1$ ($\theta_f = 0$)

so denominator is smallest.

$\therefore$ bounce straight back

16B2 Q3

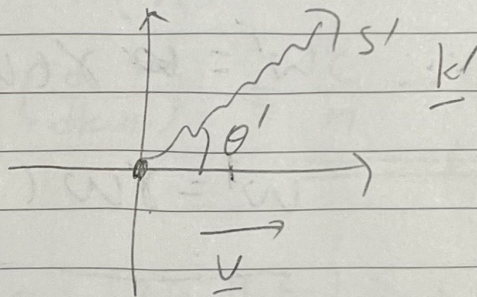
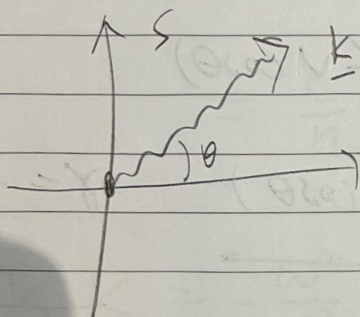
— phenomena

headlight effect

~~headlight effect~~

- doppler redshift transverse doppler
- stellar aberration (headlight effect)
- Appearance of superluminal motion

— ~~D.~~ relativistic Doppler shift is the difference in frequency between the rest frame of the source and the rest frame of the observer.



S' moving w $\underline{v}$ with respect to S

4-vectors

K : 4-wave vector of light ~~in S~~

U : 4-velocity of the source

$$K \cdot U = K' \cdot U' \quad (\text{invariant})$$

$$K' = K \text{ in } S' = \begin{pmatrix} \omega'/c \\ \underline{K'} \end{pmatrix}$$

$$U' = U \text{ in } S' = \begin{pmatrix} c \\ 0 \end{pmatrix}$$

$$\therefore K' \cdot U' = -\omega'$$

$$\underline{k} \cdot \underline{v} = \begin{pmatrix} \omega/c \\ \underline{k} \end{pmatrix} \cdot \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$$

$$= -\gamma\omega + \gamma \underline{k} \cdot \underline{v} = \cancel{-\gamma\omega} - \gamma(\omega - \underline{k} \cdot \underline{v})$$

$$= -\omega'$$

$$\therefore \omega' = \cancel{\omega} \gamma(\omega - \underline{k} \cdot \underline{v})$$

photon $\underline{k} = \frac{\omega}{c} \hat{k}$ $\hat{k} \cdot \underline{v} = v \cos \theta$

$$\therefore \omega' = \cancel{\omega} \gamma \left(\omega - \frac{\omega}{c} v \cos \theta \right)$$

$$\underline{\omega' = \gamma \omega (1 - v/c \cos \theta)} \quad \underline{\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}}$$

— ~~phase~~ phase / group velocity

$$\text{phase velocity} = v_{ph} = \frac{\omega}{|\underline{k}|} \hat{k} = \boxed{\frac{\omega}{|\underline{k}|^2} \underline{k}}$$

$$\text{group velocity} = v_{gr} = \nabla_{\underline{k}} \omega = \boxed{\frac{\partial \omega}{\partial \underline{k}}}$$

If $\underline{k}$ defined to be in x -direction

$$\underline{v_{ph} = \frac{\omega}{k}} \quad , \quad \underline{v_{gr} = \frac{d\omega}{dk}}$$

- refractive index n

$$\omega = \frac{c}{n|k|} \quad (n \text{ may be a function of } \omega/|k|)$$
$$= \frac{c}{n} k$$

$$\therefore \text{4 wave vector } \underline{K} = \begin{pmatrix} \omega/c \\ \underline{k} \end{pmatrix}$$

$$~~V_{ph}~~ \quad V_{ph} = \frac{\omega}{|k|} = \underline{\underline{\frac{c}{n}}}$$

$$V_{gr} = \frac{d\omega}{dk} = c \frac{d}{dk} \left(\frac{k}{n} \right)$$

$$= \frac{c}{n} + ck \left(\frac{d}{dk} \frac{1}{n} \right) = \underline{\underline{\frac{c}{n} - \frac{ck}{n^2} \frac{dn}{dk}}}$$

$$\underline{K} \cdot \underline{K} = -\frac{\omega^2}{c^2} + k^2 = -\left(\frac{c^2}{n^2}\right)k^2 \cdot \frac{1}{c^2} + k^2$$
$$= k^2 \left(1 - \frac{1}{n^2} \right)$$

If $n=1$, then $\underline{K} \cdot \underline{K} = 0$

$$\therefore -\frac{\omega^2}{c^2} + k^2 = 0$$

differentiate this ~~with~~ with k

$$- \frac{1}{c^2} \cdot 2\omega \frac{d\omega}{dk} + 2k = 0$$

$$\therefore \omega \frac{d\omega}{dk} = kc^2$$

$$\therefore \frac{\omega}{k} \cdot \frac{d\omega}{dk} = c^2 \Rightarrow \underline{\underline{V_{gr} V_{ph} = c^2}}$$

- e, p collision $P = \begin{pmatrix} E/c \\ P_x \\ P_y \\ P_z \end{pmatrix}$

$$P_e = \begin{pmatrix} E_e \\ P_x \\ P_y \\ P_z \end{pmatrix}$$

$$P_p = \begin{pmatrix} E_p \\ P_x \\ P_y \\ P_z \end{pmatrix}$$

$$P_e = \begin{pmatrix} \gamma_{me} E_{me} \\ \gamma_{me} m_e v_z \\ 0 \\ 0 \end{pmatrix}$$

$$P_p = \begin{pmatrix} \gamma_{mp} E_{mp} \\ -\gamma_{mp} m_p v_z \\ 0 \\ 0 \end{pmatrix}$$

$$P_{tot} = P_e + P_p = \begin{pmatrix} E_{tot} \\ P_{tot} \\ 0 \\ 0 \end{pmatrix}$$

4 - momentum conservation:

$$P_e + P_p = P_\gamma + P_m$$

$$P_m^2 = (P_e + P_p - P_\gamma)^2$$

$$\therefore -M^2 = (P_e + P_p)^2 - 2(P_e + P_p) \cdot P_\gamma + \underbrace{P_\gamma^2}_{=0}$$

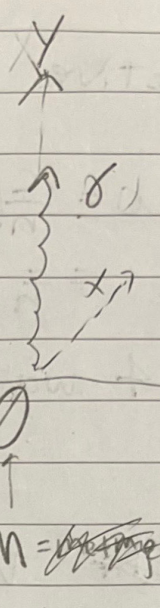
$$(P_e + P_p)^2 = -E_{tot}^2 + P_{tot}^2 = \gamma_{me}^2 E_{me}^2 -$$

$$-E_{tot}^2 + (\gamma_{me} v_z)^2 (m_p - m_e)^2$$

$$\therefore -M^2 = -E_{tot}^2 + \gamma_{me}^2 v_z^2 (m_p - m_e)^2 - 2(P_e + P_p) \cdot P_\gamma$$

$$P_e + P_p = \begin{pmatrix} E_{tot} \\ P_{tot} \\ 0 \\ 0 \end{pmatrix}, \quad P_\gamma = \begin{pmatrix} E_\gamma \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore 2(P_e + P_p) \cdot P_\gamma = +2 E_{tot} E_\gamma + 0 = -2 E_{tot} E_\gamma$$



$$P_\gamma = \begin{pmatrix} E_\gamma/c \\ 0 \\ 0 \\ 0 \end{pmatrix} = P_\gamma$$

$$E = \frac{hc}{\lambda} = \frac{2\pi\hbar}{\lambda} \quad (h = c = 1)$$

$$P_m = \begin{pmatrix} E_m \\ P_m \end{pmatrix}$$

$$= \begin{pmatrix} E_m \\ -P_m \cos \theta \\ -P_m \sin \theta \\ 0 \end{pmatrix}$$

$$\therefore -M^2 = -E_{tot}^2 + \gamma_z^2 V_z^2 (m_p - m_e)^2 + 2E_{tot} E$$

$$\therefore E_{tot}^2 - 2E_{tot} E = M^2 + \gamma_z^2 V_z^2 (m_p - m_e)^2$$

$$\therefore E_{tot}^2 - 2E_{tot} E + E^2 = M^2 + \gamma_z^2 V_z^2 (m_p - m_e)^2 + E^2$$

$$\therefore (E_{tot} - E)^2 = M^2 + \gamma_z^2 V_z^2 (m_p - m_e)^2 + E^2$$

$$\therefore -M^2 = -E$$

$$(P_e + P_p)^2 = P_{tot} \cdot P_{tot} = -(m_e + m_p)^2$$

$$(P_e \cdot P_p) \cdot P_\gamma = \begin{pmatrix} E_{tot} \\ P_{tot} \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} E \\ 0 \\ E \\ 0 \end{pmatrix}$$

$$= -E_{tot} E, \quad P_\gamma \cdot P_\gamma = 0$$

$$\therefore -M^2 = -(m_e + m_p)^2 + 2E_{tot} E$$

$$P_\gamma \cdot P_\gamma = 0$$

$$2(P_e + P_p) \cdot P_\gamma = 2 \begin{pmatrix} E_{tot} \\ P_{tot} \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} E \\ 0 \\ E \\ 0 \end{pmatrix}$$

$$= -2E_{tot} E = -2(E_e + E_p) E$$

$$(P_e + P_p)^2 = P_e^2 + P_p^2 + 2P_e \cdot P_p = -m_e^2 - m_p^2 + 2 \begin{pmatrix} E_e \\ P_e \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} E_p \\ -P_p \\ 0 \\ 0 \end{pmatrix}$$

$$= -m_e^2 - m_p^2 + 2(-E_e E_p + (P_e)(-P_p))$$

$$= -(m_e^2 + m_p^2 + 2(E_e E_p + P_e P_p))$$

$$\therefore -M^2 = -(m_e^2 + m_p^2 + 2E_e E_p + 2P_e P_p) + 2(E_e + E_p) E$$

$$\therefore E_e = \gamma_z m_e c^2, \quad E_p = \gamma_z m_p c^2$$

$$\therefore \frac{E_p}{E_e} = \frac{m_p}{m_e}$$

$$\therefore P_p = \gamma_z m_p v_z$$

$$P_e = \gamma_z m_e v_z$$

$$\gamma_z = \left(1 - \frac{v_z^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\therefore \frac{P_p}{P_e} = \frac{m_p}{m_e} \Rightarrow E_p = \frac{m_p}{m_e} E_e, \quad P_p = \frac{m_p}{m_e} P_e$$

$$E_e^2 - P_e^2 = m_e^2 \quad \therefore \cancel{E_e^2 + P_e^2 =}$$

$$\therefore P_e^2 = E_e^2 - m_e^2 \quad \therefore E_e^2 + P_e^2 = 2E_e^2 - m_e^2$$

$$\text{Similarly } \cancel{E_p^2 + P_p^2 =}$$

$$\therefore M^2 = m_e^2 + m_p^2 + 2 \frac{m_p}{m_e} (E_e^2 + P_e^2) - \cancel{2E_e^2} \frac{m_p}{m_e} - 2 \left(1 + \frac{m_p}{m_e}\right) E_e E$$

$$\therefore M^2 = m_e^2 + m_p^2 + 2 \frac{m_p}{m_e} (2E_e^2 - m_e^2) - 2 \left(1 + \frac{m_p}{m_e}\right) E_e E$$

$$M^2 = \underbrace{(m_e^2 + m_p^2 - 2m_p m_e)}_{(m_p - m_e)^2} + 4 \frac{m_p}{m_e} E_e^2 - 2E \left(1 + \frac{m_p}{m_e}\right) E_e$$

$$\therefore \cancel{M^2 = (m_p - m_e)^2}$$

$$0 = 4 \frac{m_p}{m_e} E_e^2 - 2E \left(1 + \frac{m_p}{m_e}\right) E_e + [(m_p - m_e)^2 - M^2]$$

$$\therefore E_e = \cancel{\quad}$$

$$\therefore 4m_p E_e^2 - 2E(m_p + m_e) E_e + [(m_p - m_e)^2 - M^2] m_e = 0$$

$$\therefore E_e = \frac{1}{8m_p} \left[2E(m_p + m_e) \pm \sqrt{4E^2(m_p + m_e)^2 - 4 \cdot 6m_p m_e [(m_p - m_e)^2 - M^2]} \right]^{\frac{1}{2}}$$

~~take the~~ $M = m_p + m_e$

~~inside square root~~ $\therefore (m_p - m_e)^2 - (m_p + m_e)^2 = 4m_p m_e$

$$E_{tot} = E_e + E_p = E_e \left(1 + \frac{m_p}{m_e} \right) \text{ (ignore negative root)}$$

$$\therefore E_{tot} = \frac{1}{8} \left(\frac{1}{m_e} + \frac{1}{m_p} \right) \left[2E(m_p + m_e) + \sqrt{4E^2(m_p + m_e)^2 + 64m_p^2 m_e^2} \right]$$

Where $\underline{E = \frac{hc}{\lambda}}$

(or use $M = m_p + m_e = 13.6 \text{ eV}$ as the minimum of M)

$\rightarrow$ hence the min of E_{tot}

$$E = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}mv^2 = \frac{3}{2}mv^2$$

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- EM wave

$$\underline{E} = (0, E_y, 0) \quad \underline{B} = (B_x, 0, 0), \quad \underline{k} = (0, 0, k)$$

$$E_y = E_0 \cos(\omega t - kz) \quad \cancel{E_y z}$$

$$B_x = B_0 \cos(\omega t - kz)$$

Maxwell's equations with index of refraction n .

$$(n \equiv \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}})$$

$$\therefore \frac{1}{\sqrt{\mu \epsilon}} = \frac{c}{n}$$

$$\mu \epsilon = \frac{n^2}{c^2}$$

$$\nabla \cdot \underline{E} = 0$$

$$\nabla \cdot \underline{B} = 0$$

$$\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{B} = \mu \epsilon \frac{\partial \underline{E}}{\partial t} = \frac{n^2}{c^2} \frac{\partial \underline{E}}{\partial t}$$

$$\nabla \cdot \underline{E} = \frac{\partial E_y}{\partial y} = 0 \quad \checkmark$$

$$\nabla \cdot \underline{B} = \frac{\partial B_x}{\partial x} = 0 \quad \checkmark$$

$$\nabla \times \underline{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = \hat{j} (-\partial_z E_y)$$

$$= -\hat{x} (\partial_z E_y) = -k E_0 \sin(\omega t - kz) \hat{x}$$

$$\frac{\partial \underline{B}}{\partial t} = \cancel{-\omega B_0 \cos(\omega t - kz)} - \omega B_0 \sin(\omega t - kz) \hat{x}$$

$$\therefore -kE_0 = \omega B_0$$

$$\therefore \frac{\omega}{k} = -\frac{E_0}{B_0}$$

$$\underline{\nabla} \times \underline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \partial_z \\ B_x & 0 & 0 \end{vmatrix} = -\hat{j}(0 - \partial_z B_x) \\ = \partial_z B_x \hat{j}$$

$$= +\omega B_0 k \sin(\omega t - kz) \hat{j}$$

$$\frac{n^2}{c^2} \frac{\partial E}{\partial t} = -\frac{n^2}{c^2} \omega E_0 \sin(\omega t - kz) \hat{j}$$

$$\therefore B_0 k = -\frac{n^2}{c^2} \omega E_0$$

$$\therefore \frac{n^2}{c^2} \frac{\omega}{k} = \frac{k}{\omega} \quad \therefore \left(\frac{\omega}{k}\right)^2 = \left(\frac{c}{n}\right)^2$$

$$\rightarrow \underline{\frac{\omega}{k} = \frac{c}{n}}$$

$$\therefore \underline{B} = \underline{\nabla} \times \underline{A} = (B_x, 0, 0)$$

$$= (B_x(z), 0, 0)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\therefore A_x = 0 \quad \therefore B_y = 0 \quad \text{we only need}$$

$$\underline{A} = (0, A_y, 0) \quad \underline{\nabla} \times \underline{A} = \underline{j} (-\partial_z A_y)$$

$$\therefore -\partial_z A_y = B_x = B_0 \cos(\omega t - kz)$$

$$\therefore A_y = - \int B_0 \cos(\omega t - kz) dz$$

$$= \frac{B_0}{k} \sin(\omega t - kz)$$

$$\therefore \underline{A} = \begin{pmatrix} 0 \\ \frac{B_0}{k} \sin(\omega t - kz) \\ 0 \end{pmatrix}$$

$$\frac{\partial A}{\partial t} = \frac{\omega}{k} B_0 \cos(\omega t - kz)$$

$$= -\frac{E_0}{B_0} B_0 \cos(\omega t - kz)$$

$$= -E_0 \cos(\omega t - kz)$$

$$\therefore \underline{E} = -\frac{\partial \underline{A}}{\partial t} = E_0 \cos(\omega t - kz)$$

$$\therefore \underline{E} = -\underline{\nabla} \phi - \frac{\partial \underline{A}}{\partial t} \quad \therefore \underline{\nabla} \phi = 0$$

$$\therefore \phi = \text{const} \xrightarrow{\text{set to}} 0$$

check Lorenz gauge $\partial^\mu A_\mu = 0$

$$\partial^\mu = \left(-\frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$A_\mu = (-A - \phi/c, A_x, A_y, A_z)$$

$$\therefore \nabla \cdot \underline{A} = 0 \rightarrow \frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \underline{A} = 0$$

$$\text{if } \phi = 0, \quad \underline{A} = \frac{B_0 \hat{y}}{k} \sin(\omega t - kz) \quad \checkmark$$

$$\therefore \underline{E}_0 \times \underline{B}_0 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & E_0 & 0 \\ B_0 & 0 & 0 \end{vmatrix} = -\hat{z} B_0 E_0$$

$$\therefore |\underline{E}_0 \times \underline{B}_0| = |\underline{E}_0 B_0| =$$

$$\therefore \left| \frac{E_0}{B_0} \right| = \frac{\omega}{k} = \frac{c}{n}, \quad |\underline{E}_0| |\underline{B}_0| =$$

$$\therefore \frac{n}{c} |\underline{B}_0| \frac{c}{n} B_0^2 = 1$$

$$\therefore \underline{B}_0 = \sqrt{\frac{c}{n}}, \quad \underline{E}_0 = -\frac{\omega}{k} \underline{B}_0 = -\frac{n}{c}$$

$$B_0 = \sqrt{\frac{n}{c}}, \quad E_0 = -\frac{\omega}{k} B_0 = -\frac{c}{n} \sqrt{\frac{n}{c}} = -\sqrt{\frac{c}{n}}$$

$$\therefore \underline{\phi} = 0 \quad \underline{A} = \frac{\hat{y}}{k \sqrt{n}} \sin(\omega t - kz)$$

$$\text{wave vector } \underline{k} = \begin{pmatrix} \omega/c \\ 0 \\ k \end{pmatrix}$$

$$\therefore \underline{k} = (0, 0, k) \quad \therefore \underline{k} = \begin{pmatrix} \omega/c \\ 0 \\ 0 \\ k \end{pmatrix}$$

- EM field invariant quantities

$$D = B^2 - \frac{E^2}{c^2}, \quad \alpha = \frac{\mathbf{B} \cdot \mathbf{E}}{c}$$

$\therefore \alpha = 0$, $\therefore$ it is possible for either $\underline{E}$ or $\underline{B}$ to vanish

$$D = B^2 - \frac{E^2}{c^2} = \cos^2(\omega t - kz) \left(B_0^2 - \frac{E_0^2}{c^2} \right)$$

$$= \cos^2(\omega t - kz) \left(\frac{n}{c} - \frac{1}{c} \frac{E}{n} \right)$$

$$= \frac{1}{c} \cos^2(\omega t - kz) \left(n - \frac{1}{n} \right)$$

$$\therefore n > 1 \quad \therefore D > 0$$

$\therefore$ only Electric field is possible to vanish.

Transformation of EM field

$$\underline{E}'_{\parallel} = \underline{E}_{\parallel} \quad \underline{E}'_{\perp} = \gamma(\underline{E}_{\perp} + \underline{v} \times \underline{B})$$

$$\underline{B}'_{\parallel} = \underline{B}_{\parallel} \quad \underline{B}'_{\perp} = \gamma\left(\underline{B}_{\perp} - \frac{\underline{v} \times \underline{E}}{c^2}\right)$$

$$\therefore \underline{E}' = 0 \quad \therefore \underline{E}'_{\parallel} = 0, \quad \underline{E}'_{\perp} = 0$$

$\underline{E}'_{\parallel} = 0$, need to choose a $\underline{v}$ such that

$$\underline{E}_{\parallel} = (\underline{E} \cdot \underline{\hat{v}})\underline{\hat{v}} = 0 \quad \therefore \underline{E} \perp \underline{v}$$

$$\underline{E}'_{\perp} = \gamma(\underline{E}_{\perp} + \underline{v} \times \underline{B}) = 0$$

$$\therefore \underline{E}_{\perp} = -\underline{v} \times \underline{B}$$

$$\therefore \underline{E}_{\parallel} = \underline{E}'_{\parallel} = 0 \quad \therefore \underline{E}_{\perp} = \underline{E}$$

$$\therefore \underline{E} = -\underline{v} \times \underline{B}$$

$$\therefore \underline{E} = E_y \hat{y} \quad , \quad \underline{B} = B_x \hat{x}$$

$$\therefore \underline{v} = -v \hat{z} \quad \omega$$

$$\text{so that } \cancel{\underline{z}} (-v) \hat{z} \times B_x \hat{x} = -v \cancel{\underline{z}} \times \hat{x}$$

$$= \cancel{v} \hat{z} \quad v B_x (\hat{z} \times \hat{x}) = v B_x \hat{y} = E_y \hat{y}$$

$$v = \frac{|\underline{E}|}{|\underline{B}|} \hat{z}$$

$$\therefore \underline{v} = -\frac{|\underline{E}|}{|\underline{B}|} \hat{z} \quad \underline{v} = -\frac{E_y}{B_x} \hat{z}$$

$$\underline{v} = -\frac{E_0 \cos(\omega t - kz)}{B_0 \cos(\omega t - kz)} \hat{z}$$

$$= -\frac{E_0}{B_0} \hat{z}$$

check.

In lab frame.

$$\underline{V} = -\frac{E_0}{B_0} \hat{z}$$

$$\therefore \underline{E}_{\parallel} = 0 \quad \underline{E}_{\perp} = E_y \hat{y}$$

$$\underline{B}_{\parallel} = 0 \quad \underline{B}_{\perp} = B_x \hat{x}$$

$$\underline{E}'_{\parallel} = \underline{E}_{\parallel} = 0$$

$$\underline{E}'_{\perp} = \gamma (\underline{E}_{\perp} + \underline{V} \times \underline{B})$$

$$= \gamma (E_y \hat{y} - \frac{E_0}{B_0} \hat{z} \times B_x \hat{x})$$

$$= \gamma (E_y \hat{y} - \frac{E_y}{B_x} B_x (\hat{z} \times \hat{x}))$$

$$= \hat{y} \gamma (E_y - E_y) = 0$$

✓

$$\hat{V} = -\frac{1}{B}$$

$$E_1 = E_2$$

$$B_1 = B_2$$

$$E_1 = 0, E_2 = 0$$

$$E_1 = X(E_1 + V \times B)$$

$$= X(E_1) + -\frac{1}{B} \times B \times X$$

$$= X(E_1) - \frac{1}{B} \times B \times X$$

$$= \hat{V} X(E_1 - E_1) = 0$$

UB2 Q4

$$A^\mu = \begin{pmatrix} \phi/c \\ \underline{A} \end{pmatrix} \quad \underline{E} = -\nabla\phi - \frac{\partial \underline{A}}{\partial t} \quad \underline{B} = \nabla \times \underline{A}$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$\partial^c F^{ab} + \partial^a F^{bc} + \partial^b F^{ca}$$

$$= \partial^c \partial^a A^b - \partial^c \partial^b A^a + \partial^a \partial^b A^c - \partial^a \partial^c A^b + \partial^b \partial^c A^a - \partial^b \partial^a A^c$$

$$= \underline{\underline{0}}$$

~~$$J_\mu = \rho / \rho_0$$~~

$$J_\mu = -\rho_0 \begin{pmatrix} -c \\ \underline{j} \end{pmatrix} = -\rho_0 \underline{v}$$

4-velocity.
(contravariant)

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E/c \\ E/c & 0 & B_z & -B_y \\ & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{pmatrix}$$

covariant

$$\partial^\mu = \left(-\frac{1}{c} \frac{\partial}{\partial t}, \underline{\nabla} \right)$$

$$\partial^\mu F_{\mu\nu} = \left(-\frac{1}{c} \frac{\partial}{\partial t}, \underline{\nabla} \right) \begin{pmatrix} 0 & -E/c \\ E/c & 0 & B_z & -B_y \\ & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{pmatrix}$$

$$= \left(\frac{\nabla \cdot \underline{E}}{c}, +\frac{1}{c^2} \frac{\partial E}{\partial t} - \nabla \times \underline{B} \right)$$

$$\therefore \nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0} \quad (M1)$$

$$\therefore \frac{\nabla \cdot \underline{E}}{c} = \frac{\rho}{\epsilon_0 c} = \frac{\rho c}{\epsilon_0 c^2}$$

$$\therefore c^2 = \frac{1}{\mu_0 \epsilon_0}$$

~~$$\epsilon_0 c^2 = \frac{1}{\mu_0}$$~~

$$\epsilon_0 c^2 = \frac{1}{\mu_0}$$

$$\therefore \frac{\nabla \cdot \underline{E}}{c} = \mu_0 \rho c$$

$$\therefore \nabla \times \underline{B} = \mu_0 \underline{j} + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t}$$

$$\therefore \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} - \nabla \times \underline{B} = -\mu_0 \underline{j}$$

$$\therefore \partial^\mu F_{\mu\nu} = (\mu_0 \rho c, -\mu_0 \underline{j})$$

$$= -\mu_0 (-\rho c, \underline{j}) = \underline{j}_\nu$$

$$\therefore \partial^\mu F_{\mu\nu} = j_\nu$$

Lorentz force per unit volume is

~~$$f_\nu = \rho F^{\mu\nu} U_\mu$$~~

~~$$f^\nu = \rho (\underline{E} + \underline{v} \times \underline{B})$$~~

~~$$= \rho F^{\mu\nu} U_\mu = \rho F^{\mu\nu} U_\nu$$~~

contravariant

(the contravariant version)

rest charge density
↓

$$f^\mu = \rho_0 F^{\mu\nu} U_\nu \Rightarrow f^\nu = \rho_0 F^{\nu\lambda} U_\lambda$$

turn to covariant

$$f_\mu = g_{\mu\nu} \rho_0 F^{\nu\lambda} U_\lambda$$

$$= \rho_0 F_\mu^\lambda U_\lambda$$

$$= \rho_0 g^{\lambda\alpha} F_{\mu\alpha} U^\lambda$$

$$= \rho_0 g^{\lambda\alpha} F_{\mu\alpha} g_{\lambda\beta} U^\beta = g^{\lambda\alpha} g_{\lambda\beta} F_{\mu\alpha} U^\beta$$

$$= \rho_0 \delta^\alpha_\beta F_{\mu\alpha} U^\beta = \rho_0 \delta^\alpha_\lambda \delta^\lambda_\beta F_{\mu\alpha} U^\beta$$

$$= \rho_0 F_{\mu\lambda} U^\lambda$$

$$= \rho_0 U^\lambda F_{\mu\lambda} = \rho_0 U^\nu F_{\mu\nu}$$

$$= -\cancel{J_\mu} - J^\nu F_{\mu\nu} = J^\nu (-F_{\mu\nu})$$

$$= \underline{J^\nu F_{\nu\mu}}$$

$$\begin{aligned} f_0 &= J^\nu F_{\nu 0} = \underbrace{J^0 F_{00}}_0 + \underbrace{J^1 F_{10}}_{-\vec{E}_x/c} + \underbrace{J^2 F_{20}}_{-\vec{E}_y/c} + \underbrace{J^3 F_{30}}_{-\vec{E}_z/c} \\ &= \frac{\rho_0}{c} (\vec{j} \cdot \vec{E}) \end{aligned}$$

is the rate of work done on charge per unit volume.

$$W^N = \begin{pmatrix} \underline{S} \cdot \underline{P} \\ \frac{E \underline{S}}{c} \end{pmatrix} \quad E = \gamma m_0 c^2 \quad \frac{E}{m_0 c} = \gamma c$$

$$U_w^N = \frac{W^N}{m_0 c |\underline{S}|} = \begin{pmatrix} \frac{\underline{S} \cdot \underline{P}}{m_0 c |\underline{S}|} \\ \frac{\gamma \underline{S}}{|\underline{S}|} \end{pmatrix}$$

4 momentum $P = \begin{pmatrix} E/c \\ \underline{P} \end{pmatrix} = \begin{pmatrix} \gamma m c \\ \gamma m \underline{v} \end{pmatrix}$

in rest frame, $\underline{P}' = 0$, $\underline{v}' = 0$, $\underline{S}' = \underline{S}_0$, $\gamma = 1$
(S')

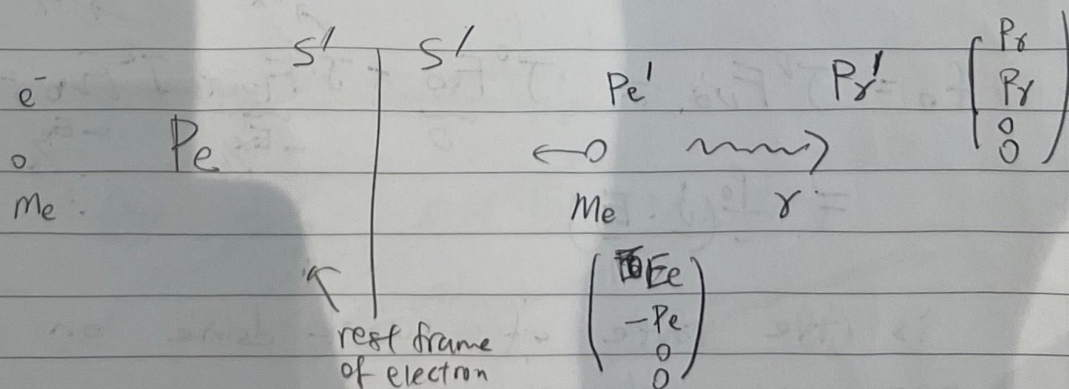
and, the value of $U_w^N P_N$ is invariant

$$\therefore U_w^N P_N = U_w'^N P_N' = \begin{pmatrix} 0 \\ \frac{\underline{S}_0}{|\underline{S}_0|} \end{pmatrix} \cdot \begin{pmatrix} mc \\ 0 \end{pmatrix}$$

$$= \underline{\underline{0}}$$

electron is an elementary particle (invariant rest mass)

ex (a) $n=1$ $\therefore$ photon speed = c



$$\text{let } c = k = 1$$

$$P_e = P_e' + P_\gamma'$$

$$P_e = \begin{pmatrix} m_e \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

photon

$$\therefore P_e' = P_\gamma' - P_e$$

$$P_\gamma = \begin{pmatrix} P_\gamma \\ P_\gamma \\ 0 \\ 0 \end{pmatrix}$$

✓ $E = P$.
if velocity $= c$

$$\therefore P_e' \cdot P_e' = (P_\gamma' - P_e)^2$$

$$P_e' = \begin{pmatrix} E_e' \\ -P_e' \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore -m_e^2 = \underbrace{P_\gamma' \cdot P_\gamma'}_0 + \underbrace{P_e \cdot P_e}_{-m_e^2} - 2P_\gamma' \cdot P_e$$

$$\therefore P_\gamma' \cdot P_e = 0$$

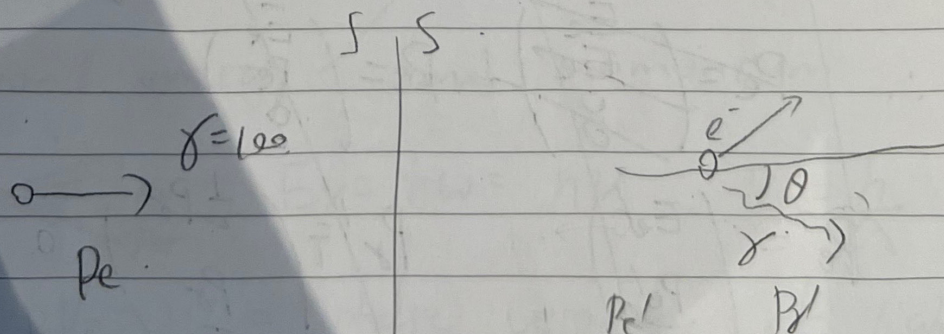
$$\therefore \begin{pmatrix} P_\gamma \\ P_\gamma \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} m_e \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\therefore P_\gamma m_e = 0 \rightarrow \underline{\underline{P_\gamma = 0}}$$

photon has zero energy

cannot ~~emit~~ emit a photon in any frame since the P_γ vector is null.

(b)



In this case the difference is that

~~$P_e = P_e' + P_\gamma$~~ velocity of ~~photon~~ photon is

$$\cancel{V_p} \quad V_p = \frac{c}{n} = \frac{c}{2}, \quad \frac{\omega}{k} = \frac{c}{n} = \frac{c}{2}$$

$$E_\gamma = \hbar \omega$$

$$P_\gamma = \hbar k$$

$$\cancel{E_\gamma = \hbar \omega}$$

$$E_\gamma = \frac{\omega}{k} P_\gamma = \frac{c}{2} P_\gamma = \frac{1}{2} P_\gamma$$

$$\cancel{P_\gamma' \cdot P_\gamma' = P_\gamma^2}$$

$$\cancel{P_\gamma}$$

$$\cancel{\left[\frac{1}{2} P_\gamma \right]}$$

$$\cancel{P_\gamma}$$

$$\cancel{\left[\frac{1}{2} P_\gamma \right]}$$

$$\cancel{= P_\gamma^2 + \frac{1}{4} P_\gamma^2 = \frac{5}{4} P_\gamma^2}$$

$$\cancel{P_\gamma^2}$$

$$\cancel{\frac{3}{4} P_\gamma^2}$$

$$P_\gamma' \cdot P_\gamma' =$$

$$\begin{pmatrix} \frac{1}{2} P_\gamma \\ P_\gamma \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{1}{2} P_\gamma \\ P_\gamma \end{pmatrix}$$

$$= -\frac{1}{4} P_\gamma^2 + P_\gamma^2 = \frac{3}{4} P_\gamma^2$$

$$P_\gamma^2$$

$$\frac{3}{4} P_\gamma^2$$

For electron

$$\gamma_e \approx 100$$

$$\therefore v_e \approx c$$

$$|P_e| \approx E_e$$

$$\therefore P_e = \begin{pmatrix} E_e \\ E_e \\ 0 \\ 0 \end{pmatrix}$$

$$P_e = \begin{pmatrix} E_e \\ P_e \\ 0 \\ 0 \end{pmatrix}$$

$$P_e' = \begin{pmatrix} E_e' \\ P_e' \end{pmatrix}$$

$$P_\gamma' = \begin{pmatrix} \frac{1}{2} P_\gamma \\ P_\gamma \end{pmatrix}$$

$$P_e = P_e' + P_r' \rightarrow P_e' = -P_e + P_r'$$

$$\cancel{-m_e^2} = \cancel{-m_e^2} + \cancel{2P_e}$$

$$\therefore \cancel{-m_e^2} = \cancel{-m_e^2} + \frac{3}{4}|P_r|^2 - 2P_e \cdot P_r'$$

$$\therefore P_r' \cdot P_e = \frac{3}{8} P_r^2 - \frac{3}{8} P_r \cdot P_r$$

$$\therefore \begin{pmatrix} E_e \\ P_e \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} P_r \\ P_r \end{pmatrix} = \frac{3}{8} P_r \cdot P_r$$

$$\therefore -\frac{1}{2} E_e P_r + P_e \cdot P_r = \frac{3}{8} P_r^2$$

$$P_e \cdot P_r = P_e P_r \cos \theta \approx E_e P_r \cos \theta$$

$\nearrow$
 $\theta = 120^\circ$

$$\therefore E_e P_r \left(\cos \theta - \frac{1}{2} \right) = \frac{3}{8} P_r^2$$

$$\therefore E_e \left(\cos \theta - \frac{1}{2} \right) = \frac{3}{8} P_r$$

photon energy and momentum

$$\cancel{E_r = \hbar \omega} \quad E_r = \hbar \omega = h\nu, \quad P_r = \frac{h}{\lambda}$$

$V_p = \lambda \nu =$ phase velocity of light.

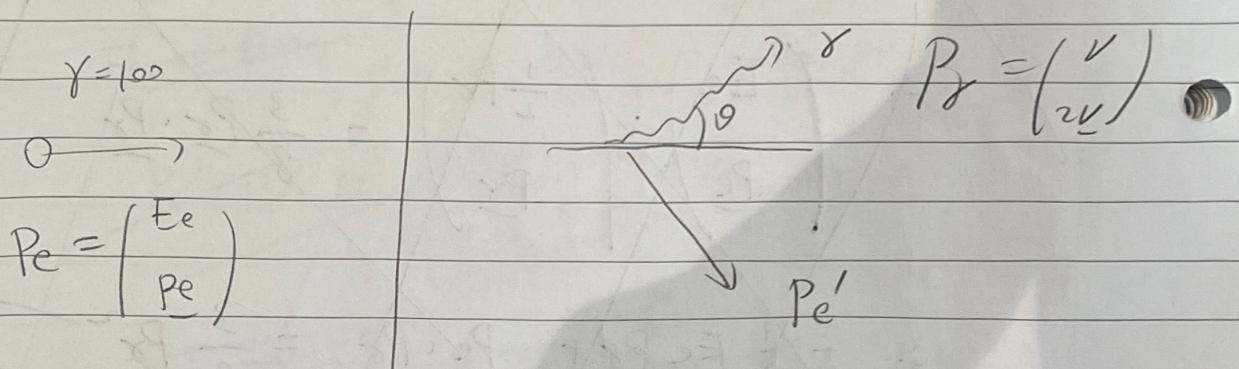
in medium $n=2$, $V_p = \frac{c}{n} = \frac{c}{2} = \frac{1}{2}c$

$$\therefore \lambda \nu = \frac{1}{2} \quad \therefore \cancel{h\nu} \quad \lambda = \frac{1}{2\nu}$$

$$\therefore P_x = \frac{h}{\lambda} = 2h\nu \quad \text{let } c = h = 1$$

$$\therefore P_x = 2\nu \quad E_x = h\nu = \nu$$

$$\therefore P_x \text{ 4vector is } P_x = \begin{pmatrix} \nu \\ 2\nu \end{pmatrix}$$



~~4-m~~ 4 momentum conservation

$$P_e = P_x' + P_e' \quad \therefore P_e' = P_e - P_x'$$

$$\therefore \underbrace{P_e' \cdot P_e'}_{-m_e^2} = \underbrace{P_e' \cdot P_e}_{-m_e^2} + P_x' \cdot P_x' - 2P_e \cdot P_x'$$

$$\therefore 2P_e \cdot P_x' = P_x' \cdot P_x'$$

$$P_e \cdot P_x' = \begin{pmatrix} E_e \\ p_e \end{pmatrix} \cdot \begin{pmatrix} \nu \\ 2\nu \end{pmatrix}$$

$$= -E_e \nu + 2\underline{p_e} \cdot \underline{\nu} = -E_e \nu + 2p_e \nu \cos \theta$$

$$p_x' \cdot p_x' = -V^2 + 4V^2 = 3V^2$$

$$\therefore 2(2p_e X \cos \theta - E_e X) = 3V^2$$

$$\therefore 2p_e \cos \theta - E_e = \frac{3V}{4}$$

$$\begin{aligned} \therefore p_e &= \gamma m_e v & E_e &= \gamma m_e c^2 \\ &= \gamma m_e c \beta & &= \gamma m_e c \\ &= \gamma m_e c \beta \end{aligned}$$

$$\therefore p_e = \beta E_e \quad \left(\beta = \frac{v_e}{c} \right) \quad \begin{array}{l} \text{velocity} \\ \text{of electron} \end{array}$$

$$\therefore E_e (2\beta \cos \theta - 1) = \frac{3V}{4}$$

$$\therefore 2\beta \cos \theta - 1 = \frac{3V}{4E_e}$$

$$\therefore \cos \theta = \frac{1}{2\beta} \left(1 + \frac{3V}{4E_e} \right)$$

$V \sim$ photon energy

— most energetic photons, gamma rays, are typically $\sim 1 \text{ MeV}$

but $E_e \sim \gamma m_e \sim 100 \times 0.5 \text{ MeV} \sim 50 \text{ MeV}$

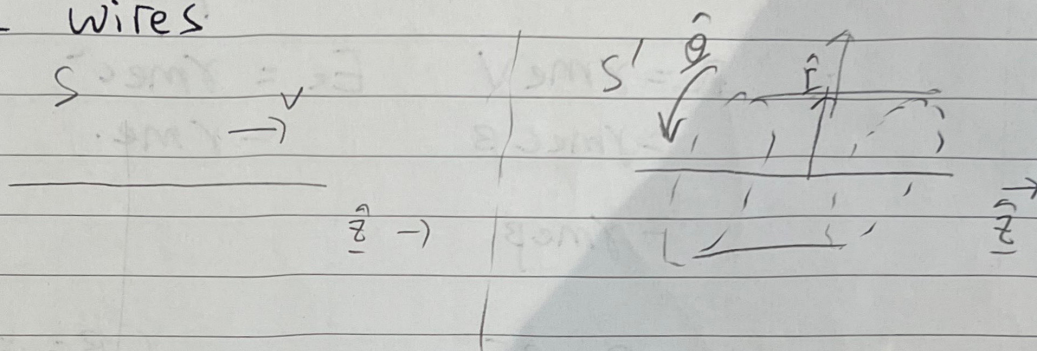
$$\therefore \frac{V}{E_e} \ll 1$$

$\therefore$ we have $\cos\theta \approx \frac{1}{\gamma\beta}$

$\therefore \gamma = 1.20 \quad \therefore \beta \approx 1$

$\therefore \cos\theta = \frac{1}{2} \rightarrow \underline{\underline{\theta = 60^\circ}}$

— the wires



(i) in S' , wire is stationary, rest charge density (ρ_0)

Use Gauss's Law $\oint \underline{E}' \cdot \underline{n} dS = \frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0} \int \rho_0 dV$

By Symmetry $\underline{E}' = E' \underline{\hat{r}}$

$(2\pi r \lambda) E' = \frac{1}{\epsilon_0} \rho \lambda$

$\therefore \rho = \gamma \rho_0$

$\therefore \underline{E}' = \frac{\rho_0 \underline{\hat{r}}}{2\pi \epsilon_0 r}$

$\therefore \underline{E}' = \frac{\rho \underline{\hat{r}}}{2\pi \epsilon_0 \gamma r}$

Stationary charges

$\underline{B}' = 0$

(ii) Lorentz transformation of EM fields

$$\underline{E}'_{\parallel} = \underline{E}_{\parallel} \quad \underline{B}'_{\parallel} = \underline{B}_{\parallel}$$

$$\underline{E}'_{\perp} = \gamma(\underline{E}_{\perp} + \underline{v} \times \underline{B}) \quad \underline{B}'_{\perp} = \gamma(\underline{B}_{\perp} - \frac{\underline{v} \times \underline{E}}{c^2})$$

$$\underline{E}_{\perp} = \gamma(\underline{E}'_{\perp} - \underline{v} \times \underline{B}')$$

$$\underline{B}_{\perp} = \gamma(\underline{B}'_{\perp} + \frac{\underline{v} \times \underline{E}'}{c^2})$$

transform from S' back to S

$$\therefore \underline{v} = v \hat{z} \quad \underline{E}' = \underline{E}'_{\perp} = \frac{\rho}{2\pi\epsilon_0 r} \hat{r}$$

$$\underline{E}'_{\parallel} = 0$$

$$\underline{B}'_{\perp} = \underline{B}'_{\parallel} = 0$$

$$\therefore \underline{v} \times \underline{E}' = 0$$

$$\underline{v} \times \underline{B}' = 0$$

$$\therefore \underline{v} \times \underline{E}' = v \hat{z} \times \frac{\rho_0}{2\pi\epsilon_0 r} \hat{r}$$

$$= \frac{\rho_0 v \hat{\theta}}{2\pi\epsilon_0 r}$$

$$\therefore \underline{E}_{\parallel} = \underline{E}'_{\parallel} = 0$$

$$\underline{E}_{\perp} = \gamma(\underline{E}'_{\perp} + \underline{v} \times \underline{B}') = \gamma \underline{E}'_{\perp}$$

$$\therefore \underline{E} = \frac{\gamma \rho_0}{2\pi\epsilon_0 r} \hat{r} \quad \therefore \underline{E} = \frac{\rho \hat{r}}{2\pi\epsilon_0 r}$$

$$\underline{B}_{||}^0 = \underline{B}_{||}' = 0$$

$$\underline{B}_{\perp} = \gamma \left(\underline{B}_{\perp}' + \frac{1}{c^2} \underline{v} \times \underline{E}' \right)$$

$$= \frac{\gamma}{c^2} \frac{\rho_0 v \hat{\theta}}{2\pi \epsilon_0 r}$$

$$\therefore \boxed{\frac{\rho v \hat{\theta}}{2\pi c^2 \epsilon_0 r} = \underline{B}_{\perp}}$$

→ ~~the~~ Invariants of EM field

$$D = B^2 - \frac{E^2}{c^2}$$

$$\alpha = \frac{\underline{E} \cdot \underline{B}}{c}$$

$$\text{in } S' \quad \underline{B}' = 0 \quad \therefore \alpha = 0$$

$$D = B^2 - \frac{E^2}{c^2} < 0$$

If we can find a ~~the~~ frame in which ~~$\underline{B}$~~ $\underline{E}$ is 0, then

$$D = B^2 - \frac{0^2}{c^2} = B^2 > 0$$

contradiction

$\therefore$ No frame has $\underline{E} = 0$

$\rightarrow$ cannot be purely magnetic.

If $D \cdot B = 0$

$$D \cdot B = |D| |B| \cos \theta$$

$$\cos \theta = \frac{D \cdot B}{|D| |B|} < 1$$

$\therefore D, B$ cannot be both
time-like