A13482W1

SECOND PUBLIC EXAMINATION

Honour School of Physics – Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B3. QUANTUM, ATOMIC AND MOLECULAR PHYSICS

TRINITY TERM 2016

Thursday, 16 June, 2.30 pm – 4.30 pm

Answer **five** *questions with at least* **one** *from each section:*

Start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

1243 1243

$\delta_s = 2.931$

1. A transition in the potassium atom ($Z = 19$) is denoted by $3d\ ^2D_{5/2} - 5p\ ^2P_{3/2}$. Explain the meaning of each part of this notation. What are the selection rules relevant to determining the allowed transitions in alkali-like atomic systems? [7]

Several series of lines are observed in the emission spectrum of the alkali-like ion Ca^+ ($Z = 20$). There is a series of doublets, including the resonance lines at wavelengths of 393.4 nm and 396.9 nm, which terminates at 104.4 nm. A second series of doublets begins with lines at 373.7 nm and 370.6 nm and terminates at about 142 nm. Draw an energy level diagram showing the first doublets of both series, and the series limits. Label the levels with appropriate quantum numbers. [9]

ground state

There is a third (triplet) series which terminates at about 121.8 nm. Three emission lines are observed at 849.8 nm, 854.2 nm and 866.2 nm. Explain the origins of these lines and add them to your energy level diagram labelled with appropriate quantum numbers. [5]

The Ca^+ ion is subjected to an external magnetic field which causes the spectral lines to split into a number of separate components. For the line at 393.4 nm determine the number of components observed perpendicular to the applied field. [4]

[The spacing of the components is not required.]

$\frac{6.63\times10^{-34}\times3\times10^{8}}{E\times(1.6\times10^{-16})} = \lambda\times10^{-9}$

$\Delta E = \frac{6.63\times10^{-34}\times3\times10^{8}}{1.6\times10^{-16}\times10^{-9}}$

$1s^2 2s^2 2p^6 3s^2 3p^6 4s^1$

4s.4P

3d.

4P

$^1P_0$

$S=1\ L=1$

$^3P_{2,1,0}$

$4s^2$

$S=0$

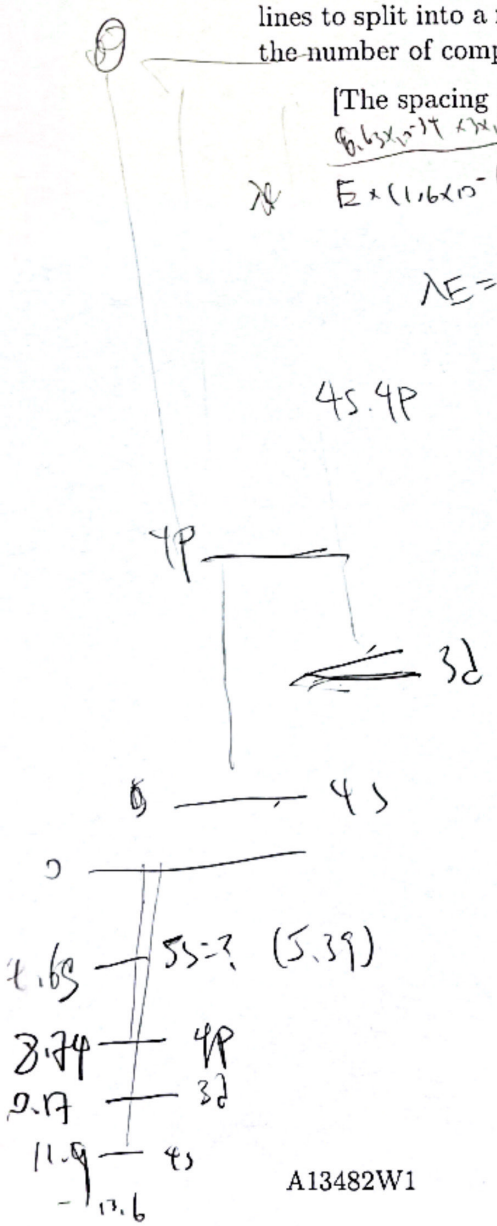
$L=0$

$J=0$

$L=1$

$S=\frac{1}{2}\ L=1$

4s4P



As Z → big, LS breaks down.
interval is not obeyed.

2. Show that an interaction with a Hamiltonian of the form $H = A\,\mathbf{J}_1 \cdot \mathbf{J}_2$ leads to an interval rule

$$E_J - E_{J-1} = AJ\ ,$$

where $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}$ are angular momentum operators related by $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, and E_J is an energy eigenvalue. [4]

What conditions must be fulfilled by the various interactions that determine atomic structure in order for an atom with two valence electrons to exhibit an interval rule? Give an example of a term in such an atom in which you would expect the rule to be obeyed. Draw a diagram showing the splitting of the term and label the levels with appropriate quantum numbers. [7]

nsnp'
SL=1
Point

The Hamiltonian for the magnetic dipole hyperfine interaction has the form

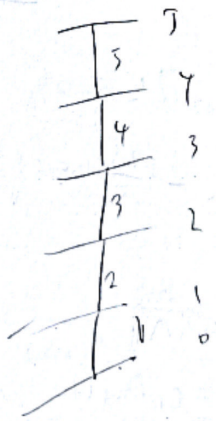
$$H_{\text{hfs}} = -g_I \mu_{\text{N}} \mathbf{I} \cdot \mathbf{B}_{\text{e}}\ ,$$

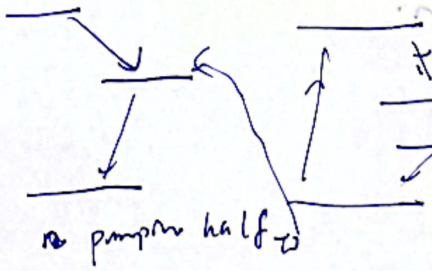
where g_I is the nuclear g-factor, $\mathbf{I}$ is the nuclear angular momentum operator and $\mathbf{B}_{\text{e}}$ is the magnetic induction at the nucleus ($r = 0$) arising from the electrons. Explain why this can be written in the form $H_{\text{hfs}} = A\,\mathbf{I} \cdot \mathbf{J}$. What set of good quantum numbers characterise the energy levels arising from H_{hfs}? [5]

Measurement of the splittings between adjacent energy levels of the ground electronic configuration in the isotope ^{55}Mn gave the following 5 frequencies: 72, 145, 217.3, 289.7, 362.1 MHz. Describe how these frequencies can be interpreted and determine the values of the properties of the atomic system that can be deduced from the data.

Measurement of the splittings between adjacent energy levels of the ground electronic configuration in the isotope ^{56}Mn gave the following 5 frequencies: 85, 141, 197.4, 253.8, 310.2 MHz. Find the ratio of the g-factors of the nuclei in the two isotopes, explaining your reasoning. [9]

5, 5, 4, 5, 3, 5, 2, 5





Ruby (3) Nd YAG laser.

3. Explain what is meant by the terms *three-level laser* and *four-level laser*, giving an example of each. Discuss why there is a large difference in the threshold power which is required to achieve laser oscillation in these two classes of laser. [5]

A homogeneously broadened transition of angular frequency ω_{21} occurs between two levels which have degeneracies g_1 and g_2. The upper level is populated at a rate R_2 and decays only by spontaneous emission to the lower level at a rate A_{21}. The lower level is populated only by this decay and has a lifetime τ_1. The number densities for the lower and upper levels are N_1 and N_2 respectively. Write down the rate equations governing N_1 and N_2 and show that the population inversion density $N^* = N_2 - (g_2/g_1)N_1$ in the steady state is given by

$\frac{dI}{dz} = N^* \sigma I.$

$$N^* = R_2\left(\frac{1}{A_{21}} - \frac{g_2}{g_1}\tau_1\right).$$

$\dot{N}_2 = R_2 - N_2 A_{21}$

$\dot{N}_1 = N_2 A_{21} - \frac{N_1}{\tau_1}$

$N^* = N_2 - \frac{g_2}{g_1}N_1 = \frac{R_2}{A_{21}}\left(1 - \frac{g_2}{g_1}A_{21}\tau_1\right).$ [4]

In the presence of radiation of intensity I (at angular frequency ω), stimulated processes transfer atoms between the levels. Write down the rate equations governing N_1 and N_2 for this system, defining additional quantities that you introduce. Show that the gain coefficient at line centre has the form

$1 - \frac{g_2}{g_1}A_{21}\tau_1 > 0.$

$$\alpha = \frac{\alpha_0}{1 + I/I_s}.$$

$\frac{dI}{dz} = \frac{\alpha}{1 + I/I_s} I = \alpha I_s$ linear growth.

Give expressions for α_0 and I_s, and state what they represent. For the case $I \gg I_s$, explain how the intensity changes as radiation propagates through the medium. [12]

A homogeneously broadened laser transition has a peak optical cross-section $\sigma(\omega_{21}) = 4 \times 10^{-15}\,\mathrm{cm}^2$, and $R_2 = 5 \times 10^{19}\,\mathrm{s}^{-1}\,\mathrm{cm}^{-3}$. The upper level has decay rate $A_{21} = 3 \times 10^6\,\mathrm{s}^{-1}$ and $g_2 = 5$. The lower level has lifetime $\tau_1 = 2 \times 10^{-8}\,\mathrm{s}$ and $g_1 = 3$. Determine the value of α_0. $0.06/\mathrm{cm} > 0.$

Another transition has a comparable cross-section and its upper level is pumped at the same rate. In this case $A_{21} = 3 \times 10^6\,\mathrm{s}^{-1}$, $g_2 = 3$, $\tau_1 = 3 \times 10^{-7}\,\mathrm{s}$ and $g_1 = 1$. What is the value of α_0 for this transition? [4]

< 0

$\frac{dN_2}{dt} = R_2 - A_{21}N_2 = 0$

$\frac{dN_1}{dt} = A_{21}N_2 - \frac{N_1}{\tau_1} = 0$

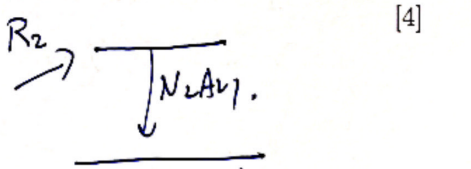
$\therefore N_2 = \frac{R_2}{A_{21}}$

$N_1 = \tau_1 A_{21} N_2$

$= \tau_1 A_{21}\frac{R_2}{A_{21}} = \tau_1 R_2$

$= \frac{\frac{R_2}{A_{21}}\left(1 - \frac{g_2}{g_1}A_{21}\tau_1\right)}{1 + \frac{\sigma I}{\hbar\omega A_{21}}}.$

$\therefore N^* =$



$\dot{N}_2 = R_2 - N_2A_{21} - \frac{\sigma I}{\hbar\omega}\left(N_2 - \frac{g_2}{g_1}N_1\right)$

$\dot{N}_1 = N_2A_{21} - \frac{N_2}{\tau_2} + \frac{\sigma I}{\hbar\omega}\left(N_2 - \frac{g_2}{g_1}N_1\right).$

$\dot{N}_1 + \dot{N}_2 = R_2 - \frac{N_1}{\tau_1} = 0.$

$N^* = N_2 - \frac{g_2}{g_1}N_1 = R_2\tau_1\frac{\left(\frac{1}{\tau_1} - \frac{g_2}{g_1}A_{21}\right)}{A_{21} + \frac{\sigma I}{\hbar\omega}}.$

$\alpha_0 = \frac{R_2}{A_{21}}\left(1 - \frac{g_2}{g_1}A_{21}\tau_1\right)$

$I_s = \frac{\hbar\omega A_{21}}{\sigma}.$

4. In a two-state model of an atom there are two eigenstates, ψ_1 and ψ_2, of the Hamiltonian $\hat{H}_0$ with energies E_1 and E_2 respectively ($E_2 > E_1$). Spontaneous emission is neglected. This system is subjected to a perturbation such that

$$\hat{H} = \hat{H}_0(\mathbf{r}) + \hat{V}(t),$$

where the time-dependent perturbation $\hat{V}(t) = ex\mathcal{E}\cos\omega t$ arises from an oscillating electric field of amplitude $\mathcal{E}$ along the x-direction. The general solution of this time-dependent Schrödinger equation can be written

$$\Psi(\mathbf{r},t) = c_1(t)\psi_1(\mathbf{r})\mathrm{e}^{-\mathrm{i}E_1t/\hbar} + c_2(t)\psi_2(\mathbf{r})\mathrm{e}^{-\mathrm{i}E_2t/\hbar}\,.$$

Show that

$$\dot{c}_2(t) = -\mathrm{i}\Omega\cos(\omega t)\mathrm{e}^{\mathrm{i}\omega_0 t}\,c_1(t) - \frac{\mathrm{i}V_{22}}{\hbar}\,c_2(t)\,,$$

giving the expressions for ω_0, V_{22}, and the Rabi frequency, Ω, in terms of matrix elements of the atomic dipole (such as $\langle\psi_1|ex|\psi_2\rangle$) and other quantities. Explain why $V_{22} = 0$. [10]

The initial condition is $c_2(0) = 0$. Assuming that $c_2(t) \ll c_1(t)$ and $c_1(t) \simeq 1$, calculate the integral

$$c_2(t) = \int_0^t \dot{c}_2(t')\mathrm{d}t'\,.$$

[5]

Justify the use of the *rotating wave approximation* and find an expression for $|c_2(t)|^2$. [4]

The above model applies for monochromatic radiation. Show how it can be extended to describe an atom illuminated by weak broadband radiation of spectral energy density $\rho(\omega)$ which is related to the electric field by

$$\rho(\omega)\,\mathrm{d}\omega = \frac{1}{2}\epsilon_0\mathcal{E}^2\,.$$

Hence find an expression for the transition rate due to stimulated absorption of broadband radiation. [6]

board.

$$\frac{e^2}{2\epsilon_0\hbar^2}\left|\langle\psi_1|x|\psi_2\rangle\right|^2 \int \propto \int \left(\frac{\sin((\omega-\omega_0)t/2)}{\sin(\omega_0-\omega)}\right)^2 \rho(\omega_0)\,d\omega$$

$$= 2\pi t\,\rho(\omega_0)$$

[You may find this integral useful:

$$\int_{-\infty}^{\infty}\left(\frac{\sin\frac{1}{2}(\omega-\omega_0)t}{\frac{1}{2}(\omega-\omega_0)}\right)^2 \mathrm{d}\omega = 2\pi t\,.]$$

$$|c_2(t)|^2 = \frac{e^2}{2\epsilon_0\hbar^2}\left|\langle\psi_2|x|\psi_1\rangle\right|^2\rho(\omega_0)$$

narrow

$$R_{12} = \frac{d|c(t)|^2}{dt} = \frac{|c(t)|^2}{t} = \underbrace{\frac{\pi e^2}{\epsilon_0\hbar^2}\left|\langle\psi_2|x|\psi_1\rangle\right|^2}_{B_{21}}\rho(\omega_0)$$

16B3 Q1

$3d^1$ $^2D_{5/2}$ $5p$ $^2P_{3/2}$

$n=3$ $l=2$ $L=2$ $J=\frac{5}{2}$ $S=\frac{1}{2}$

$n=5$ $l=1$ $L=1$ $J=\frac{3}{2}$ $S=\frac{1}{2}$

n = priciple quantum number

l = angular momentum quantum of ~~vat~~ single electron (orbital)

L = total ~~orbi~~ orbital angular momentum

S = total ~~spin of a~~ electronic spin of atom

J = total electronic angular momentum in atom

in this case K is a alkali-mental $\therefore l=L$ $\because$ only one valence electron.

- Selection rules :

$\Delta J = 0, \pm 1$ ($J=0 \not\rightarrow 0$)

$\Delta L = \Delta l = \pm 1$ (No $\Delta L = 0$ $\because$ only one electron ~~det~~ that determines L $\therefore L=l$).

parity change

$\Delta S = 0$ $S=\frac{1}{2}$

Δn = anything

3/7

- Spectral lines of Ca^+

energy and wavelength conversion

$$E = \frac{hc}{\lambda} \quad \therefore \quad E\,(1.6\times10^{-19}) = \frac{(6.63\times10^{-34})(3\times10^{8})}{\lambda(10^{-9})}$$

$$\therefore \quad \lambda E = \frac{6.63\times10^{-34}\times3\times10^{8}}{1.6\times10^{-19}\times10^{-9}} = 1243.13$$

$\lambda E = 1243.13$ ~~for~~ λ in nm
E in eV

Convert all λ into E:

~~fo~~ Doublet 1 :

393.4 nm → $\Delta E_1 =$ ~~3.16eV~~ 3.160 eV
396.9 nm → $\Delta E_2 =$ ~~3.13eV~~ 3.132 eV

104.4 nm → $E_0 = 11.91$ eV

Doublet 2 :

373.7 nm → ~~$\Delta E = 3.33$ eV~~
$\Delta E_1' = 3.327$ eV
370.6 nm → $\Delta E_2' = 3.354$ eV

142 nm → $E' = 8.754$ eV

can only be caused by P–S transition ($\Delta l = 0, \pm 1$) $0 \not\to 0$

$\because$ only S is unsplit. (e.g. P–D, both spit, X)

The Doublet is due to spin-orbit interaction

in S state, $s = \frac{1}{2}$, $l = 0$ $\therefore j = \frac{1}{2}$

in P state, $s = \frac{1}{2}$, $l = 1$ $\therefore j = \underbrace{\frac{3}{2}, \frac{1}{2}}_{\text{doublet}}$

$\because Ca^+ (Z=20)$ $\therefore$ ground state ~~$1s^2 4s$~~ $\underline{1s^2 2s^2 2p^6 3s^2 3p^6 4s}$

Doublet 1 is from $\underline{4p\ ^2P_{\frac{3}{2},\frac{1}{2}}}$ to $4s\ ^2S_{\frac{1}{2}}$

~~($\because$ 104.4 has larger energy than 142, so 104.4 can reach ground state)~~

resonant line means ground to first excited

Doublet 2 is from $\underline{5p\ ^2S_{\frac{1}{2}}}$ to $4p\ ^2P_{\frac{3}{2},\frac{1}{2}}$

this is because the spliting in this two Doublets is the same

$\frac{1}{104.4} = \frac{1}{396.9} + \frac{1}{142}$

$$\delta = |\Delta E_1 - \Delta E_2| = 0.28\,eV$$

$$\delta' = |\Delta E_1' - \Delta E_2'| = 0.27\,eV$$

$$\delta \sim \delta'$$

$\therefore$ The state that is split must be the same (in this case 4p)

For the triplet state

$849.8\,nm \to \Delta E_1'' = 1.463\,eV$

$854.2\,nm \to \Delta E_2'' = 1.455\,eV$

$866.2\,nm \to \Delta E_3'' = 1.435\,eV$

121.8 nm → ~~E'~~ $E'' = 10.21$ eV

∵ E'' is between E' and E

∴ lowest state of this triplet Transition is in between 4s and 4p,

→ this must be 3d

∴ We have from ~~4p~~ 4p to 3d

triplets are $^2P_{\frac{3}{2}}$ to $^2D_{\frac{5}{2}}$

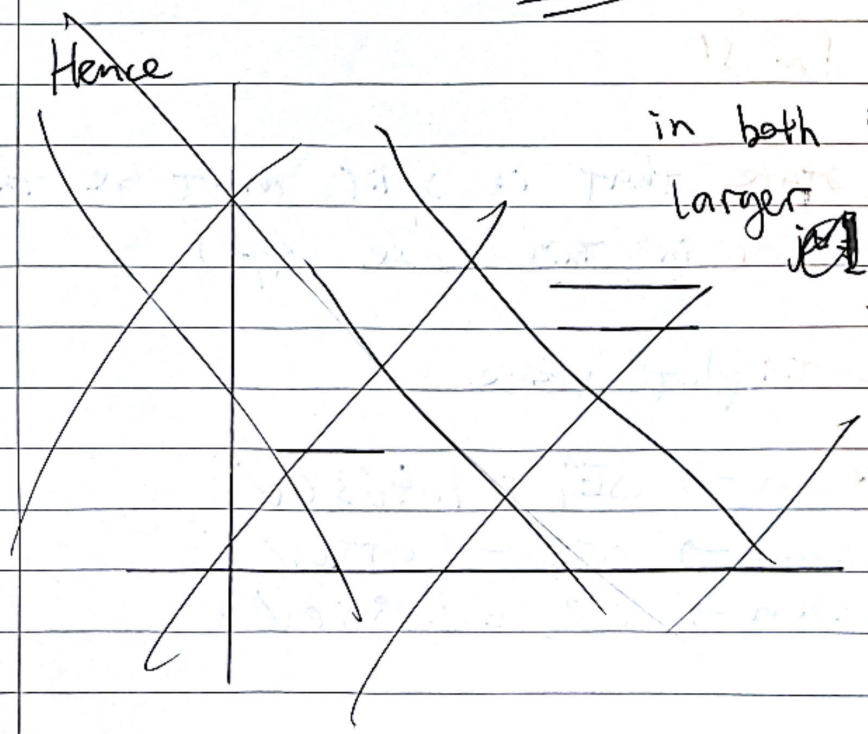
$^2P_{\frac{3}{2}}$ to $^2D_{\frac{3}{2}}$

$^2P_{\frac{1}{2}}$ to $^2D_{\frac{3}{2}}$

$^2P_{\frac{1}{2}}$ to $^2D_{\frac{5}{2}}$ is forbidden, because

$\Delta j = 0, \pm 1$, $\Delta j \neq -2$ in dipole transition.

Hence



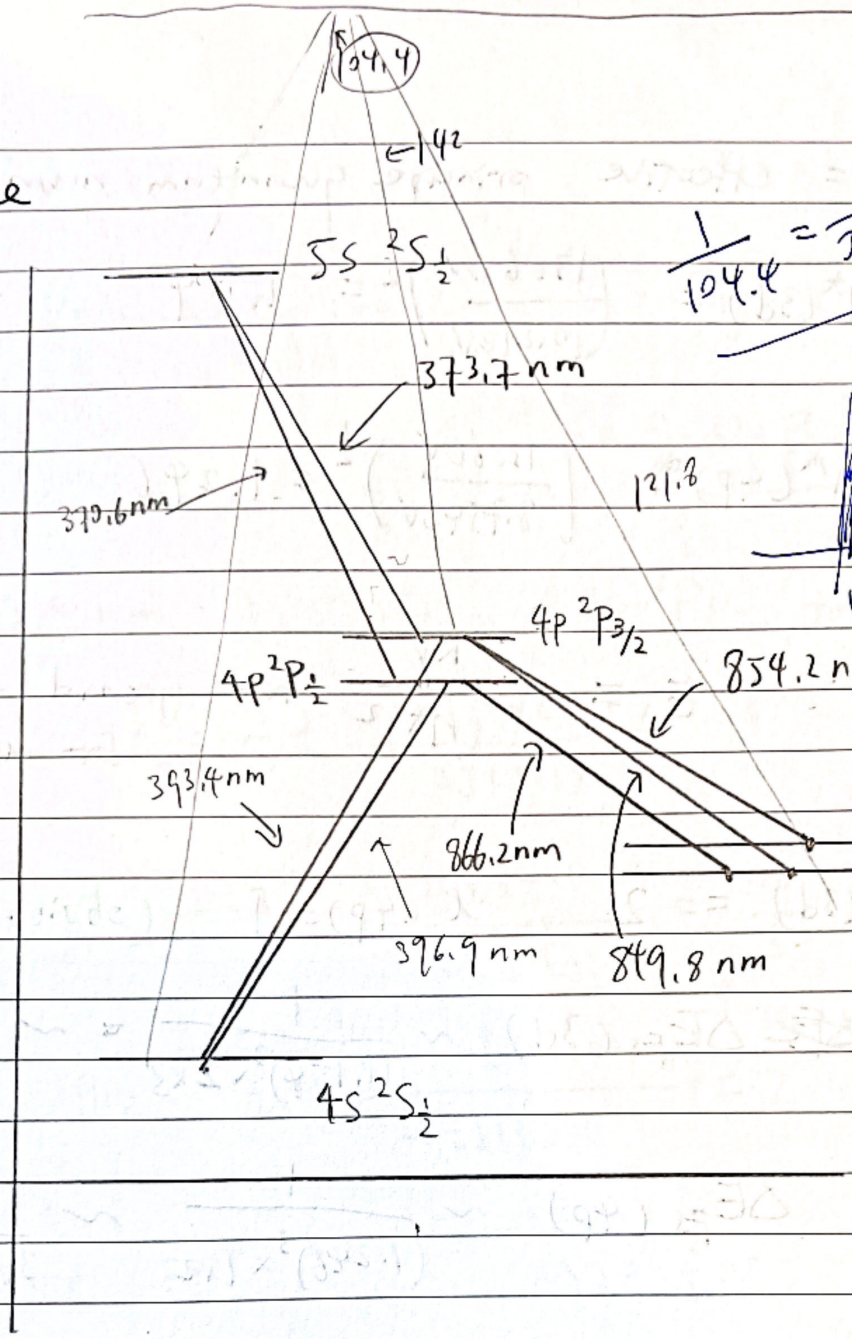
in both 3d, 4p,

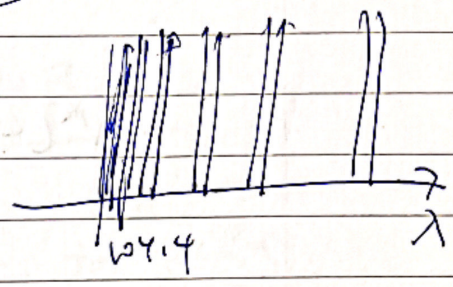
larger j is higher than smaller j

∵ less than half filled

Hence

$$\frac{1}{104.4} = \frac{1}{79.9} + \frac{1}{142}$$





Within the triplet, $^2P_{3/2}$ to $^2D_{3/2}$ has to have largest energy → 849.8 nm

Now, which ~~is~~ of $^2P_{3/2}$ to $^2D_{5/2}$, and $^2P_{1/2}$ to $^2D_{3/2}$, has ~~me~~ larger energy shift?

- Fine structure splitting with a term ~~is~~ for alkali ~~metal is~~ ion is

$$\Delta E_{FS} = \frac{Z_i^2 Z_0^2 \alpha^2}{(n^*)^3 l(l+1)} Ry \propto \frac{1}{(n^*)^3 l(l+1)}$$

n^* = effective principe quantum number

$$n^*(3d) = \left(\frac{13.6\,eV}{10.21\,eV}\right)^{\frac{1}{2}} = 1.154$$

$$n^*(4p) = \left(\frac{13.6\,eV}{8.754\,eV}\right)^{\frac{1}{2}} = 1.246$$

$$\left(\because \quad E = -\frac{Ry}{(n^*)^2} \right.$$ for ground states for alkali)

$l(3d) = 2$, $l(4p) = 1$ (obviously)

$$\therefore \quad \cancel{\Delta E_{FS}}\ \Delta E_{FS}(3d) \sim \frac{1}{(1.154)^3 \times 2 \times 3} \sim \frac{1}{9.22}$$

$$\Delta E_{FS}(4p) \sim \frac{1}{(1.246)^3 \times 1 \times 2} \sim \frac{1}{3.865}$$

$\therefore \Delta E_{FS}(4p) > \Delta E_{FS}(3d)$

$\therefore$ $^2P_{3/2}$ to $^2D_{5/2}$ has larger energy shift than $^2P_{1/2}$ to $^2D_{3/2}$

— Zeeman effect.

Zeeman splitting $E_J = g_J \mu_B B \hat{J}_z$

$$= g_J \mu_B B M_J$$

393.4 nm is from $4p\ ^2P_{3/2}$ to $4s\ ^2S_{1/2}$

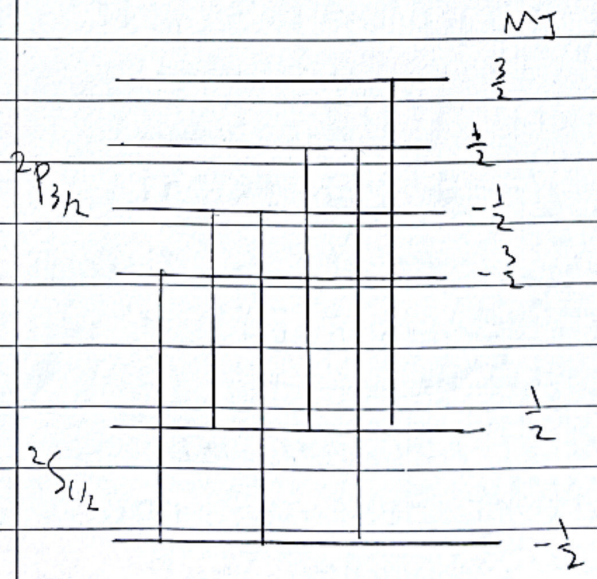
$$\therefore \quad g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$$

$$\therefore g_{3/2} = \frac{3}{2} + \frac{(\frac{1}{2})(\frac{3}{2}) - (1)(2)}{2(\frac{3}{2})(\frac{5}{2})} = \frac{4}{3} \checkmark$$

$$g_{1/2} = \frac{3}{2} + \frac{(\frac{1}{2})(\frac{3}{2}) - (0)(1)}{2(\frac{1}{2})(\frac{3}{2})} = 2 \checkmark$$

For ~~$J=\frac{3}{2}$~~ $^2P_{3/2}$, $M_J = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$

For $^2S_{1/2}$, $M_J = \frac{1}{2}, -\frac{1}{2}$



Selection rule

$\Delta M_J = 0, \pm 1$

$(\Delta M_J \neq 0$ if $J = 0)$

$\therefore$ 4 σ transitions

$(\Delta M_J = \pm 1)$

2 π transitions

$(\Delta M_J = 0)$

viewed from ~~both~~ perpendicular to $\underline{B}$ field

→ both σ and π can be seen

∴ can see $\underline{\underline{6}}$ components.

~~16B2Q2~~ 16B3Q2

- interval rule

$$H = A\,\underline{J_1}\cdot\underline{J_2}$$

Define $\underline{J} = \underline{J_1} + \underline{J_2}$

$\therefore J^2 = J_1^2 + J_2^2 + 2\underline{J_1}\cdot\underline{J_2}$

$\therefore \underline{J_1}\cdot\underline{J_2} = \frac{1}{2}(J^2 - J_1^2 - J_2^2)$

$\therefore H = \frac{A}{2}(J^2 - J_1^2 - J_2^2)$

Eigenstates $|J_1 J_2 J M_J\rangle$

Eigenenergies $E_J = \frac{A}{2}(J(J+1) - J_1(J_1+1) - J_2(J_2+1))$

For same J_1, J_2, $J = |J_1 - J_2|, \cdots, J_1 + J_2$

$E_J - E_{J-1} = \frac{A}{2}(J(J+1) - (J-1)J)$

$= \frac{A}{2} J(\cancel{J}+1 - \cancel{J}+1) = \frac{A}{\cancel{2}} \cdot J \cdot \cancel{2} = \underline{\underline{AJ}}$

- in atom, $\underline{J_1} = \underline{L}$, $\underline{J_2} = \underline{S}$. for interval rule to be obeyed, we need to be in the <u>LS coupling</u> scheme

this requires the energy splitting due to spin-orbit interaction is much less than the residual electrostatic interaction that gives rise to the terms.

- Example Ca atom

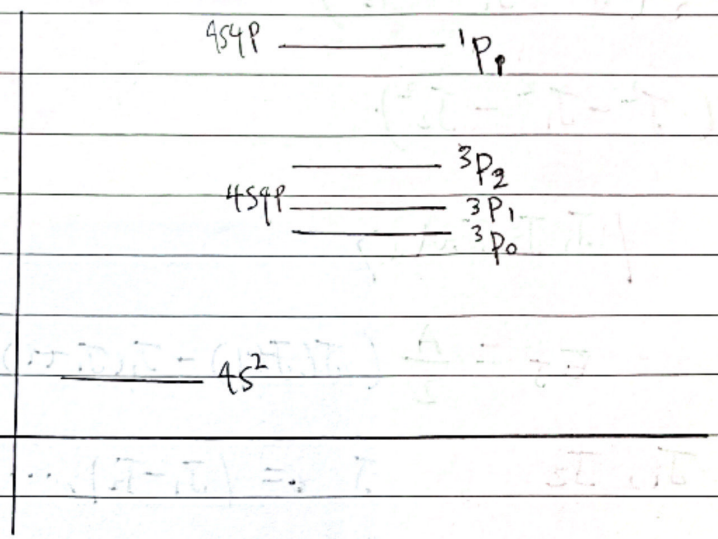
~~[Ar] 4s²~~ Ground state $[Ar]\,4s^2\ {}^1S_0$

First excited state $[Ar]\,4s4p$

consider the triplet $S = \frac{1}{2} + \frac{1}{2} = 1$

$L = 0 + 1 = 1$

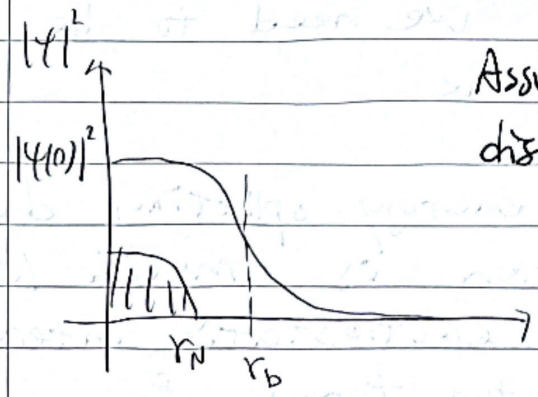
$J = 2, 1, 0$



- Hyperfine structure

the magnetic Be created by electrons on the nucleus is :

for S electron



Assume a uniform charge distribution with $a_0 \gg r_b \gg r_N$

radius of nucleus

Magnetisation in the ~~sphe~~ sphere is

$$\underline{M} = -g_s \mu_B \underline{S} |\psi(\underline{r})|^2$$

Assume in $r < r_b$, $\psi(\underline{r}) = \psi(0)$

$$\therefore \underline{M} = -g_s \mu_B |\psi(0)|^2 \underline{S}$$

~~B_e~~

- charge distribution outside r_b produces no magnetic field at $r=0$

- inside r_b. $\because$ a sphere of uniform magnetisation

$$\therefore \underline{B_e} = \frac{2}{3}\mu_0 \underline{M}$$

$$\therefore B_e = -\frac{2}{3}\mu_0 g_s \mu_B |\psi(0)|^2 \underline{S} \quad \propto -\underline{S}$$

$$H_{HFS} = -g_I \mu_N \underline{I} \cdot \underline{B} = \frac{2}{3} \underbrace{\mu_0 \mu_N g_I g_s \mu_B |\psi(0)|^2}_{A} \underline{I} \cdot \underline{S}$$

$\because S$ electron $\therefore \underline{J} = \underline{S}$ $\therefore \underline{I} \cdot \underline{S} = \underline{I} \cdot \underline{J}$

$$\therefore \underline{H_{HFS} = A \underline{I} \cdot \underline{J}}$$

For electron ~~with~~ with $l \neq 0$.

$$\underline{B_e} = \frac{\mu_0}{4\pi}\left\{\frac{(-e)\underline{r} \times (-\underline{v})}{r^3} - \frac{\underline{\mu_e} - 3(\underline{\mu_e} \cdot \underline{r})\underline{r}}{r^3}\right\}$$

$$= 2\frac{\mu_0}{4\pi}\frac{\mu_B}{r^3}\left\{\underline{1} - \underline{S} + \frac{3(\underline{S} \cdot \underline{r})\underline{r}}{r^2}\right\}$$

$(\mu_e = \cancel{-2\mu} - g_s \mu_B \underline{S} \simeq -2\mu_B \underline{S})$

~~So In LS coupling:~~

$\underline{l}$ and $\underline{S}$ precesses around $\underline{J}$, so

all components perpendicular to $\underline{J}$

average to 0 $\therefore$ $\underline{B_e}$ or $\underline{J}$

$\therefore$ $\underline{H_{hfs} = A\,\underline{I}\cdot\underline{J}}$

- let $\underline{F} = \underline{I} + \underline{J}$ = total angular momentum of entire atom.

then good quantum numbers are

F, M_J, I, J (n, L, S)

$\because$ $H_{hfs} = A\,\underline{I}\cdot\underline{J} = \frac{A}{2}[\hat{F}^2 - \hat{I}^2 - \hat{J}^2]$

$E_{hfs} = \frac{A}{2}(F(F+1) - I(I+1) - J(J+1)]$

- By interval rule, for the same electronic state (fixed I, J)

$E_F - E_{F-1} = AF$

$\therefore E_{F-1} - E_{F-2} = A(F-1)$

$$\therefore \quad X = \frac{E_F - E_{F-1}}{E_{F-1} - E_{F-2}} = \frac{F}{F-1}$$

$$\therefore \quad XF - X = F \quad \therefore \quad F = \frac{X}{X-1}$$

	ΔE_F	X	F
362.1	—	—	
289.7	72.4	—	
217.3	72.4	1	2
145	72.3	~1	2
72	73	~1	2

$\therefore F = 2$ for ground state of ^{55}Mn

$\therefore$ 5 lines $\therefore$

ΔE_F	X	F
$E_5 - E_4$ 362.1	—	—
$E_4 - E_3$ 289.7	1.25	5
$E_3 - E_2$ 217.3	1.33	4
$E_2 - E_1$ 145	1.5	3
$E_1 - E_0$ 72	2	2

$\therefore F = 0, 1, 2, 3, 4, 5$

$\because$ F can be 0 $\therefore$ $I = J = \frac{5}{2}$

— ^{56}Mn (same electronic configuration ∵ ~~same~~ isotope)

ΔE_F	X	F
$E_{5.5}-E_{4.5}$ 310.2	—	—
$E_{4.5}-E_{3.5}$ 253.8	1.22	5.5
$E_{3.5}-E_{2.5}$ 197.4	1.286	4.5
$E_{2.5}-E_{1.5}$ 141	1.4	3.5
$E_{1.5}-E_{0.5}$ 85	1.659	2.5

∴ F = 5.5, 4.5, 3.5, 2.5, 1.5, 0.5

∵ $J = \frac{5}{2}$ ∴ $I = 3$

from ^{55}Mn

$$E^{56}_{5.5} - E^{56}_{4.5} = A_{56}(5.5) = 310.2$$

$$E^{55}_{5} - E^{55}_{4} = A_{55}(5) = 362.1$$

$$\therefore \frac{A_{56}}{A_{55}} = \frac{310.2 \times 5}{362.1 \times 5.5} = 0.779$$

∵ $A_{56} \propto g_{56}$ $A_{55} \propto g_{55}$

∴ the required ratio is 0.779.

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- three level laser / 4 level laser:

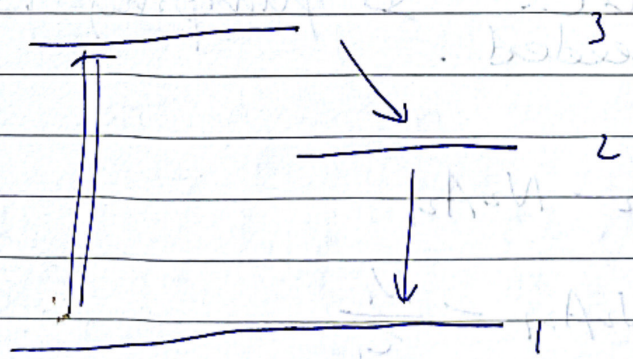
population inversion cannot be achieved by only two energy levels, we need more levels to have $N^* > 0$.

population inversion is achieved by pumping from lowest level to highest level ($0 \to 3$)

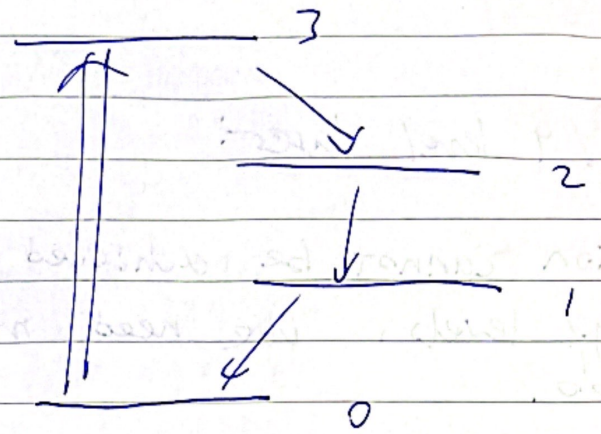
level 3 rapidly decays non-radiatively (by collision e.g.) to level 2.

lasing occurs between $2 \to 1$ because a population inversion ~~as~~ has been created between 2 and 1.

If the lowest level lies close to ($E_1 \ll k_B T$) or is the ground level, then the laser is a 3-level laser (0 is 1)



If the lowest level is well above ($E_1 \gg k_B T$) the ground state, then it is a 4-level laser



Threshold power needed for lasing is much lower for 4-level laser than for 3-level laser because for 4 level laser, level 1 can drain its population into level 0, so that the condition for population inversion, $N_2 - \frac{g_2}{g_1} N_1 > 0$ is easier to be achieved. For 3 level laser, however, The level 1 is the ground state, there ~~are~~ is a large population in level 1 that cannot decay to any where. So pumping is much more needed.

$$\frac{dN_2}{dt} = R_2 - N_2 A_{21}$$

$$\frac{dN_1}{dt} = N_2 A_{21} - \frac{N_1}{\tau_1}$$

in steady state, $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$

$\therefore R_2 = N_2 A_{21}$

$N_2 A_{21} = \frac{N_1}{\tau_1}$ $\qquad \therefore N_2 = \frac{R_2}{A_{21}}$

$\frac{R_2 \cancel{A_{21}}}{\cancel{A_{21}}} = \frac{N_1}{\tau_1}$

$\therefore N_1 = R_2 \tau_1$

$$\therefore N^* = N_2 - \frac{g_2}{g_1} N_1$$

$$= \frac{R_2}{A_{21}} - \frac{g_2}{g_1} R_2 \tau_1$$

$$= R_2 \left(\frac{1}{A_{21}} - \frac{g_2}{g_1} \tau_1 \right)$$

define $\sigma_{21}(\omega - \omega_{21}) = \frac{\hbar\omega}{c} B_{21} g_H(\omega - \omega_{21})$

cross section. $\qquad$ homogeneous broadening ~~beardening~~ Lorentzian.

The rate equations

$$\frac{dN_2}{dt} = R_2 - N_2 A_{21} - \frac{\sigma I}{\hbar\omega} N^*$$

$$\frac{dN_1}{dt} = \cancel{R_1 + N_2} N_2 A_{21} + \frac{\sigma I}{\hbar\omega} N^* - \frac{N_1}{\tau_1}$$

Steady state $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$

$$R_2 - N_2 A_{21} - \frac{\sigma I}{\hbar\omega} N^* = 0 \qquad (1)$$

$$N_2 A_{21} + \frac{\sigma I}{\hbar\omega} N^* - \frac{N_1}{\tau_1} = 0 \qquad (2)$$

$$(1) + (2) \Rightarrow \quad R_2 - \frac{N_1}{\tau_1} = 0 \quad \therefore N_1 = R_2 \tau_1$$

$$\therefore N_2 A_{21} + \frac{\sigma I N^*}{\hbar\omega} - R_2 = 0$$

$$\therefore N_2 = -\frac{1}{A_{21}} \frac{\sigma I N^*}{\hbar\omega} + \frac{R_2}{A_{21}}$$

$$N^* = N_2 - \frac{g_2}{g_1} N_1$$

$$= -\frac{1}{A_{21}} \frac{\sigma I N^*}{\hbar\omega} + \frac{R_2}{A_{21}} - \frac{g_2}{g_1} R_2 \tau_1$$

$$\therefore N^* \left(1 + \frac{1}{A_{21}} \frac{\sigma I}{\hbar\omega}\right) = R_2 \left(\frac{1}{A_{21}} - \frac{g_2}{g_1} \tau_1\right)$$

$$\therefore N^* = \frac{R_2 \left(\frac{1}{A_{21}} - \frac{g_2}{g_1} \tau_1\right)}{1 + I \Big/ \left(\frac{\hbar\omega A_{21}}{\sigma}\right)}$$

gain coefficient

$$\alpha = N^* \sigma = \frac{R_2 \sigma \left(\frac{1}{A_{21}} - \frac{g_2}{g_1} \tau_1\right)}{1 + \frac{I}{\left(\frac{\hbar\omega A_{21}}{\sigma}\right)}}$$

$$\equiv \frac{\alpha_0}{1 + I/I_s}$$

For $I \gg I_s$, $\frac{I}{I_s} \gg 1$

$$\therefore \alpha \simeq \frac{\alpha_0 I_s}{I}$$

radiation propagates along z

$$\frac{dI}{dz} = \alpha I \quad \therefore \frac{dI}{dz} \simeq \frac{\alpha_0 I_s \cancel{I}}{\cancel{I}} = \alpha_0 I_s$$

$$\therefore I = I_0 + \alpha_0 I_s z$$

intensity increases linearly.

— numbers.

$$\alpha_0 = R_2 \sigma \left(\frac{1}{A_{21}} - \frac{g_2}{g_1} \tau_1 \right)$$

(i) $\alpha_0 = 0.06 \text{ cm}^{-1}$

(ii) $\alpha_0 = \cancel{-0.13} \; -0.113 \text{ cm}^{-1}$ < 0

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$$\hat{H} = \hat{H}_0(\underline{r}) + \hat{V}(t)$$

$$\hat{V}(t) = ex\,\mathcal{E}\cos\omega t$$

$$\Psi(\underline{r},t) = c_1(t)\psi_1(\underline{r})e^{-iE_1t/\hbar} + c_2(t)\psi_2(\underline{r})e^{-iE_2t/\hbar}$$

Schrodinger equation $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$

$$\therefore\ i\hbar\frac{\partial}{\partial t}\left(c_1\psi_1 e^{-iE_1t/\hbar} + c_2\psi_2 e^{-iE_2t/\hbar}\right)$$

$$= i\hbar\left[\dot{c}_1\psi_1 e^{-iE_1t/\hbar} - i\frac{E_1}{\hbar}\cancel{c_1\psi_1 e^{-iE_1t/\hbar}} + \dot{c}_2\psi_2 e^{-iE_2t/\hbar} - i\frac{E_2}{\hbar}\cancel{c_2\psi_2 e^{-iE_2t/\hbar}}\right]$$

$$= \hat{H}\Psi = H_0\Psi + V\Psi$$

$$= E_1c_1\cancel{\psi_1 e^{-iE_1t/\hbar}} + E_2c_2\cancel{\psi_2 e^{-iE_2t/\hbar}}$$

$H\psi_i = E_i\psi_i$

$$+ Vc_1\psi_1 e^{-iE_1t/\hbar} + Vc_2\psi_2 e^{-iE_2t/\hbar}$$

$$\therefore\ i\hbar\dot{c}_1\psi_1 e^{-iE_1t/\hbar} + i\hbar\dot{c}_2\psi_2 e^{-iE_2t/\hbar} = Vc_1\psi_1 e^{-iE_1t/\hbar} + Vc_2\psi_2 e^{-iE_2t/\hbar}$$

multiply above by ψ_2^* and integrate over whole space. $\Rightarrow$

$\because \int \psi_2^* \psi_1 \, d^3\underline{r} = 0 \qquad \int \psi_2^* \psi_2 \, d^3\underline{r} = 1$

(orthogonality)

$\therefore$

$$i\hbar \dot{c}_2 e^{-iE_2 t/\hbar} = c_1 \langle \psi_2 | V | \psi_1 \rangle e^{-iE_1 t/\hbar}$$

$$+ c_2 \langle \psi_2 | V | \psi_2 \rangle e^{-iE_2 t/\hbar}$$

define matrix elements $\langle \psi_2 | V | \psi_1 \rangle \equiv V_{21}$

$\langle \psi_2 | V | \psi_2 \rangle \equiv V_{22}$

$$\therefore i\hbar \dot{c}_2 = c_1 V_{21} e^{i\frac{(E_2 - E_1)}{\hbar} t} + c_2 V_{22}$$

multiply by $\frac{-i}{\hbar}$

$$\Rightarrow \quad \dot{c}_2 = -\frac{i}{\hbar} c_1 V_{21} e^{i\omega_0 t} - \frac{i}{\hbar} V_{22} c_2$$

$$\left(\omega_0 \equiv \frac{E_2 - E_1}{\hbar} \right)$$

$$V_{21} = \int \psi_2^* V \psi_1 \, d^3\underline{r} = \int \psi_2^* e x \mathcal{E} \cos\omega t \, \psi_1 \, d^3\underline{r}$$

$$= \cancel{e\cos} \, \mathcal{E} \cos(\omega t) \int \psi_2^* e x \psi_1 \, d^3\underline{r}$$

$$= \mathcal{E} \cos(\omega t) \langle \psi_1 | e x | \psi_2 \rangle^*$$

$\because$ ~~ψ_1, ψ_2 both real (atomic wave function)~~

for bound states $\langle \psi_1 | e x | \psi_2 \rangle^*$ is real

So $V_{21} = \mathcal{E}\cos(\omega t)\langle\psi_1|ex|\psi_2\rangle$

$\therefore$ ~~$\dot{c}_2 = -\frac{i}{\hbar}\mathcal{E}\cos\omega t$~~

$$\dot{c}_2 = -i\cos(\omega t)\left[\frac{\mathcal{E}\langle\psi_1|ex|\psi_2\rangle}{\hbar}\right]e^{i\omega_0 t}c_1$$

$$\Omega \equiv \frac{\mathcal{E}\langle\psi_1|ex|\psi_2\rangle}{\hbar} \qquad -\frac{iV_{22}}{\hbar}c_2$$

$$\therefore \dot{c}_2 = -i\Omega\cos\omega t\, e^{i\omega_0 t}c_1(t) - \frac{iV_{22}}{\hbar}c_2(t)$$

$V_{22} = 0$ because in $\langle\psi_2|V|\psi_2\rangle$, atomic states have definite parity $\therefore$ $|\psi_2|^2$ must be even in parity

$$V_{22} \propto \int \psi_2^* ex\, \psi_2\, d^3r = \int \underbrace{|\psi_2|^2}_{\text{even}}\underbrace{ex}_{\text{odd}}\, d^3r = 0$$

$$\rightarrow \quad \dot{c}_2 = -i\Omega\cos(\omega t)e^{i\omega_0 t}c_1(t)$$

integral $\quad c_2(t) = \int_0^t \dot{c}_2(t')\,dt'$

$c_1 \simeq 1$

$$\therefore c_2(t) = \int_0^t -\frac{i\Omega}{2}\left(e^{i\omega t'} + e^{-i\omega t'}\right)e^{i\omega_0 t'}\,dt'$$

$$= -\frac{i\Omega}{2}\left[\int_0^t e^{i(\omega_0+\omega)t'}dt' + \int_0^t e^{i(\omega_0-\omega)t'}dt'\right].$$

$$= -\frac{i\Omega}{2}\left[\frac{1}{i(\omega_0+\omega)}\left(e^{i(\omega_0+\omega)t}-1\right)\right.$$

$$\left.+\frac{1}{i(\omega_0-\omega)}\left(e^{i(\omega_0-\omega)t}-1\right)\right]$$

$$\cancel{= -\frac{i\Omega}{2}} = \frac{\Omega}{2}\left[\frac{1-e^{i(\omega_0+\omega)t}}{\omega_0+\omega}+\frac{1-e^{i(\omega_0-\omega)t}}{\omega_0-\omega}\right]$$

- rotating wave approximation

$\omega_0+\omega \gg \omega_0-\omega$ for $\omega_0 \approx \omega$

$\therefore$ neglect ~~to~~ the term ~~ino~~ involving

$$\frac{1}{\omega_0+\omega}$$

$$C_2 \simeq \frac{\Omega}{2}\left[\frac{1-e^{i(\omega_0-\omega)t}}{\omega_0-\omega}\right]$$

$$\cancel{= -\frac{\Omega}{2}\left[1-e^{i(\omega_0-\omega)}\right.}$$

$$= \frac{\Omega}{2}\left[e^{-\frac{i(\omega_0-\omega)t}{2}} - e^{i(\omega_0-\omega)t/2}\right]$$

$$\times\left[e^{\frac{i(\omega_0-\omega)t}{2}}\right]\frac{1}{\omega_0-\omega}$$

$$= \frac{\Omega}{\cancel{2}} \cdot (-\cancel{2}i) \frac{\sin\left(\frac{\omega_0-\omega}{2} t\right)}{\omega_0-\omega} e^{i\frac{(\omega_0-\omega)t}{2}}$$

$$= -\frac{i\Omega}{2} \frac{\sin\left(\frac{\omega-\omega_0}{2} t\right)}{\frac{\omega-\omega_0}{2}} e^{-i\frac{(\omega-\omega_0)t}{2}}$$

$$\therefore |C_2(t)|^2 = \left(\frac{\Omega}{2}\right)^2 \left| \frac{\sin\left(\frac{\omega-\omega_0 t}{2}\right)}{\frac{\omega-\omega_0}{2}} \right|^2$$

Broad-band radiation

$$\Omega^2 = \frac{\mathcal{E}^2 |\langle \psi_1 | ex | \psi_2 \rangle|^2}{\hbar^2}$$

$$\therefore \rho(\omega)\, d\omega = \frac{1}{2} \varepsilon_0 \mathcal{E}^2$$

$$\therefore \Omega^2 = \frac{2|\langle \psi_1 | ex | \psi_2 \rangle|^2}{\varepsilon_0 \hbar^2} \rho(\omega)\, d\omega$$

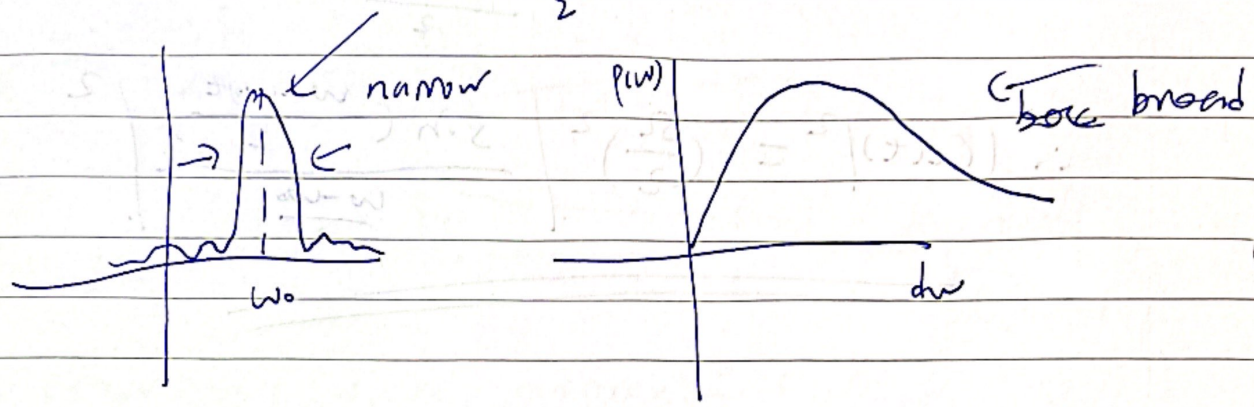
$\therefore$ total ~~[illegible]~~ $|\text{amplitude}|^2$

$$|C_2(t)|^2 = \frac{2|\langle \psi_1 | ex | \psi_2 \rangle|^2}{4\varepsilon_0 \hbar^2} \int_{-\infty}^{\infty} \rho(\omega) \left(\frac{\sin\left(\frac{1}{2}(\omega-\omega_0)\right)}{\frac{1}{2}(\omega-\omega_0)} \right)^2 d\omega$$

$$\therefore \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{1}{2}(\omega-\omega_0)\right)}{\frac{1}{2}(\omega-\omega_0)} \right)^2 d\omega = 2\pi t$$

And that the response of the atom is narrow compared ~~to~~ with the broad-band radiation.

So ① $\left(\dfrac{\sin(\frac{1}{2}(\omega-\omega_0))}{\frac{1}{2}(\omega-\omega_0)}\right)^2 \simeq 2\pi t\,\delta(\omega-\omega_0)$



$$\therefore \quad |C_2(t)|^2 = \frac{|\langle\psi_1|ex|\psi_2\rangle|^2}{2\epsilon_0\hbar^2}\int 2\pi t\,\delta(\omega-\omega_0)\rho(\omega)\,d\omega$$

$$= \frac{\pi|\langle\psi_1|ex|\psi_2\rangle|^2}{\epsilon_0\hbar^2}\rho(\omega_0)\,t$$

rate of transition R_{12} is

$$R_{12} = \frac{d|C_2(t)|}{dt} = \frac{\pi|\langle\psi_1|ex|\psi_2\rangle|^2}{\epsilon_0\hbar^2}\rho(\omega_0)$$

B_{21}