A13481W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B2: SYMMETRY AND RELATIVITY

TRINITY TERM 2015

Wednesday, 17 June, 2.30 pm – 4.30 pm

10 minutes reading time

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

$\omega' = -$

$-\gamma\beta\omega + \gamma\omega\cos\theta = \gamma\omega\cos\theta' - \gamma\beta\omega\cos\theta\cos\theta'$

$\therefore \cos\theta' = \dfrac{\gamma\cos\theta - \gamma\beta}{\gamma - \gamma\beta\cos\theta} = \dfrac{\cos\theta - \beta}{1 - \beta\cos\theta}$

Larmor's formula for the power emitted by an accelerating point charge q is

$$P_{\mathrm{L}} = \frac{\mu_0 q^2 a_0^2}{6\pi c} ,$$

where a_0 is the proper acceleration and the other symbols have their usual meaning.

$K' = \Omega K$

1. A source moving through a laboratory with speed v emits a photon at angle θ to the direction of travel. Find a formula relating θ to the direction of emission in the rest frame of the source. Show that, for a source emitting photons isotropically in its rest frame, half the photons are emitted into a cone with half-angle $\cos^{-1}(v/c)$ in the laboratory. [6]

Show that, for a body whose acceleration $\mathbf{a}$ is perpendicular to its velocity $\mathbf{v}$ in a given frame, the proper acceleration is $\gamma^2 a$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. [4]

A short bunch of electrons in a circular storage ring is used as a source of synchrotron radiation. The ring's radius is 561 m, and the energy of each electron is 3 GeV.

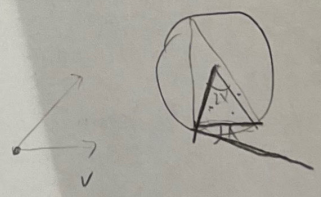
(i) Find the radiative loss per revolution for each electron.

(ii) Estimate the angular divergence of the emitted beam.

(iii) Taking into account the fact that the source is moving, estimate the pulse-length for pulses received by a distant detector at a location fixed relative to the ring.

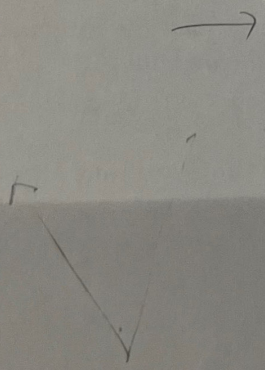
(iv) Show that for a synchrotron of given radius, the spectrum of radiation received by a detector as in (iii) has a frequency spread proportional to γ^3 when the particle speed approaches the speed of light.

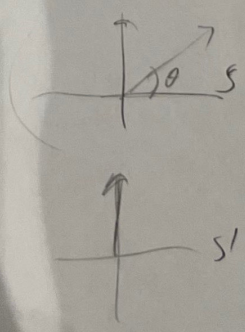
(v) In what part of the electromagnetic spectrum does the radiation mainly lie for the storage ring described? [15]



v

$\cos\theta = \frac{v}{c}$





$\sin\theta = \frac{v}{c}$

$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\gamma}$

$\frac{\gamma(1-\frac{v^2}{c^2})}{} = 1$

$= \frac{1}{\gamma^2(1-\frac{v^2}{c^2})}$

$\begin{pmatrix} 0 \\ \underline{s} \end{pmatrix}$ $\quad s^2 = \underline{s}\cdot\underline{s}$

$\frac{d}{d\tau}(s_\lambda s^\lambda) = s_a \frac{ds^a}{d\tau}$

2. Let f be a function defined over space and time, and let $\mathrm{d}f/\mathrm{d}\tau$ refer to the rate of change of f, evaluated with respect to a worldline along which the elapsed proper time is τ. Show that

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = u^\lambda \partial_\lambda f \ ,$$

$\frac{df}{d\tau} = \frac{\partial f}{\partial}$

where u^λ is the 4-velocity.

A manifestly covariant set of Euler-Lagrange equations may be written

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}}{\partial u^a}\right) = \left(\frac{\partial \mathcal{L}}{\partial x^a}\right) \ ,$$

$\frac{\partial \mathcal{L}}{\partial u^a}$ $\quad \frac{\partial(u^\lambda A_\lambda)}{du^a}$

where x^a is the 4-position. In the case of motion of a point charge q of rest mass m in an electromagnetic field, the Lagrangian $\mathcal{L} = -mc\,(-u^\lambda u_\lambda)^{1/2} + q\,u^\lambda A_\lambda$, where A^a is the 4-vector potential. Derive the equation of particle motion

$= \cancel{u^a A_a}\ q A_a$

$$m\frac{\mathrm{d}u_a}{\mathrm{d}\tau} = qF_{a\lambda}u^\lambda \ ,$$

$\frac{\partial \mathcal{L}}{\partial x^a} = 0$ $\quad \frac{\partial}{\partial u^a}(-u^\lambda u_\lambda)^{1/2}$

$= \frac{-1}{2}(-u^\lambda u_\lambda)^{-\frac{1}{2}}\, u_a$

and give an expression for F^{ab} in terms of A^a . [8]

The spin 4-vector s^α has components $(\gamma\mathbf{s}\cdot\mathbf{v}/c, \gamma\mathbf{s})$ for a particle whose velocity is $\mathbf{v}$ and intrinsic angular momentum is $\mathbf{s}$ in a given frame, where $\gamma = (1 - \mathbf{v}^2/c^2)^{-1/2}$. In the absence of an applied torque, the spin 4-vector of an accelerating particle evolves as

$\frac{1}{c}$

$$\frac{\mathrm{d}s^\alpha}{\mathrm{d}\tau} = \frac{s_\lambda \dot{u}^\lambda}{c^2}\, u^\alpha \ ,$$

$u^a u_a + u^b u_b + u^c u_c + u^d u_d$

$\frac{d\underline{s}}{d\tau} = \frac{\gamma^3}{c^2}(\underline{s}\cdot\dot{\underline{v}})\,\underline{v}$ $\quad -\frac{1}{2c}u_a$

where the dot signifies $\mathrm{d}/\mathrm{d}\tau$.

(i) Show that the 4-spin remains orthogonal to the 4-velocity.

(ii) Show that the 3-spin $\mathbf{s}$ evolves as

$\dot{s}\cdot u + \dot{u}\cdot s = 0$

$$\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}\tau} = \frac{\gamma^2}{c^2}\left[\,(\mathbf{s}\cdot\dot{\mathbf{v}})\mathbf{v} - (\mathbf{v}\cdot\dot{\mathbf{v}})\mathbf{s}\,\right] \ .$$

$\dot{u}$

(iii) Find $\mathbf{s}(\tau)$ for a particle accelerated along a straight line with speed $v(\tau) = c\,[1 - \exp(-2\,\Gamma\tau)]^{1/2}$, where Γ is constant. [11]

The spin 4-vector of a charged particle of rest mass m and charge q moving in an electromagnetic field F evolves as

$$\frac{\mathrm{d}s^a}{\mathrm{d}\tau} = \frac{gq}{2m}\left(F^{a\lambda}s_\lambda + \frac{1}{c^2}F^{\mu\nu}u_\mu s_\nu u^a\right) + \frac{s_\lambda \dot{u}^\lambda}{c^2}\,u^a$$

where g is gyromagnetic ratio. Show that, if the particle's acceleration is due to the electromagnetic force (the Lorentz force), then the magnitude of the proper spin remains constant. [6]

$\begin{pmatrix} c \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ s \end{pmatrix}$ $\quad F^{a\lambda} s_\lambda$

$P_1 + P_2 + \cdots P_N = P_{tot} = \begin{pmatrix} E_{tot} \\ \underline{P_{tot}} \end{pmatrix}$

P_{tot} $\quad -P_{tot} \cdot P_{tot} = E_{tot}^2 - P_{tot}^2$

$= (\Sigma E_i)^2 - (\Sigma |P_i|)^2$

3. A high-energy plasma contains a large number of particles. The ith particle has energy E_i and 3-momentum $\mathbf{p}_i$.

(a) Under what conditions is the quantity $\sum_i (E_i^2 - p_i^2 c^2)$, summed over half of the particles, Lorentz-invariant?

(b) Under what conditions is the quantity $(\sum_i E_i)^2 - (\sum_i \mathbf{p}_i)^2 c^2$, summed over half of the particles, Lorentz-invariant? [4]

A neutral Kaon of rest mass M has energy E in the laboratory. It decays into a pair of charged pions, such that the first pion travels, in the laboratory, in a direction perpendicular to the initial velocity of the Kaon. Show that the momenta p_1, p_2 of the pions are related to that of the Kaon, p, by

$$p^2 = p_2^2 - p_1^2 \ .$$

Show that the energy of the second pion, in the laboratory, is given by

$$E_2 = E - \frac{M^2 c^4}{2E} \ .$$

Hence find the direction of travel of the second pion, and the momenta of both pions, when $E = 600$ MeV. [11]

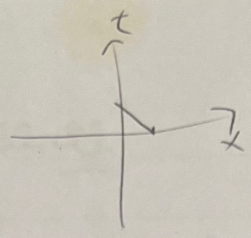
It happens that the first pion survives for a proper time of 15 ns, and the second for a proper time of 30 ns. Find the distance and time between the pion decay events in the laboratory. What is the distance between the decay events in the frame in which they are simultaneous? [10]

[The rest mass energies are: K^0: 497.6 MeV, π^+: 139.6 MeV.]

$\frac{dt}{d\tau} = \gamma$

$t' = \gamma(t - \frac{vx}{c^2})$

$\Delta t' = \gamma(\Delta t - \frac{v\Delta x}{c^2})$



$x^2 = a^2 + c^2t^2$

$t^* = \sqrt{\frac{x^2-a^2}{c^2}}$

$a=1$
$L=1$
$c=1$

4. A pair of powerful rockets accelerate continuously along a line, such that the distance between them remains constant when observed in a fixed inertial frame S. Without detailed calculation, describe qualitatively what happens to a string extended between the rockets and attached to both. [3]

Next suppose that the rockets move in such a way that the proper distance L between them remains constant. They move along the x axis of S, with rocket H always at larger x than rocket F. Let the trajectory of rocket F, relative to S, be given by $(x, y, z) = (f(t), 0, 0)$, where f is a known function. The trajectory of rocket H is then $(h(t), 0, 0)$, where the function $h(t)$ is to be discovered.

(i) Show that an event B at rocket H that is simultaneous with a given event A at rocket F, in the instantaneous rest frame of the latter, satisfies

$$c^2(t_B - t_A) = v\,[h(t_B) - f(t_A)]\ ,$$

$\int_0^t \cancel{f(t)} + ct = L$

where $v = \mathrm{d}f/\mathrm{d}t$ evaluated at t_A and all the distances and times are relative to S.

(ii) Write down, in terms of h, f, t_B, t_A and L, the condition that the proper distance between the rockets remains equal to L.

(iii) Hence show that h is given by

$$h(t + \gamma L v/c^2) = f(t) + \gamma L\ ,$$

making clear in your answer at what event the quantities v and γ are to be evaluated (in this formula). [9]

For the case $f(t) = (a^2 + c^2t^2)^{1/2}$, where a is constant, find a simple expression for γv in terms of t, and hence, or otherwise, find an explicit expression for $h(t)$.

At $t = 0$ a pulse of light is emitted from rocket H and travels towards rocket F. Find the time and place of arrival of the pulse, and the speed of F, relative to S, when the pulse arrives at F. Show that the relationship between the frequencies ω of such a pulse, as observed in the two rockets, is

$$\frac{\omega_F}{\omega_H} = \frac{a+L}{a}\ ,$$

$f(t) + ct = L$

$\sqrt{a^2+c^2t^2}^{1/2} = L - ct$

where it is assumed that each local observer agrees with what is observed by an inertial observer at rest in the instantaneous rest frame (you may assume the formula for the Doppler effect). [13]

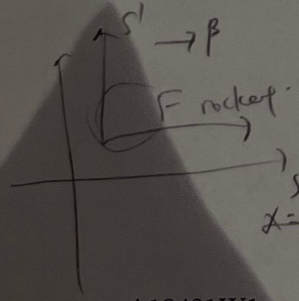
A in S: $(t_A, f(t_A))$

B in S: $(t_B, h(t_B))$

$t' = \gamma(t - \frac{vx}{c^2})$ $\quad ct' = \gamma(ct - \beta x)$

$x' = \gamma(x - vt)$ $\quad x' = \gamma(x - \beta ct)$

$t_A' = t_B'$



$\cancel{x' = \gamma(f(t_A)} -$

$a^2 + \cancel{c^2t^2} = L^2 + \cancel{c^2t^2} - 2Lct$

$v = \frac{df}{dt}$ $\quad x' = \gamma_v(f(t_A) - vt_A)$

$\therefore 2Lct = L^2 - a^2$

$\cancel{x = f(t)}$

$t_A' = \gamma_v(t_A - \frac{v}{c^2}f(t_A))$

$t_B' = \gamma_v(t_B - \frac{v}{c^2}h(t_B))$

$\boxed{t = \frac{L^2 - a^2}{2Lc}}$

$\therefore t_A - \frac{v}{c^2}f(t_A) = t_B - \frac{v}{c^2}h(t_B)$

$\frac{(L+a)^2(L-a)^2}{4L^2c^2}$

$\therefore c^2(t_B - t_A) = v(h(t_B) - h(t_A))$

a^2+

15B2Q1

The

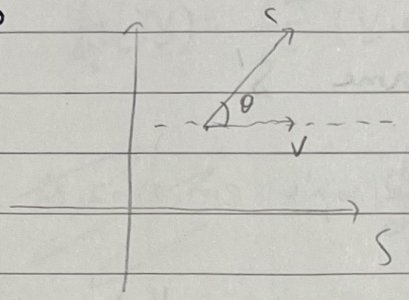
15B2 Q1

The wave 4-vector is $K = \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix}$

S' frame = rest frame

S frame = lab frame

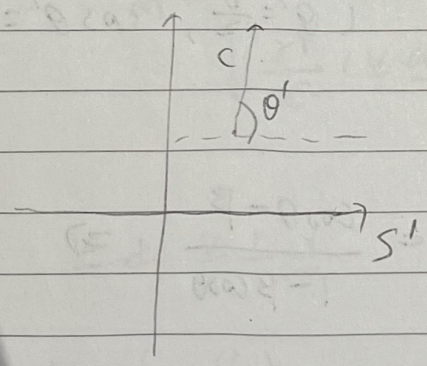
in S



$\because$ photon $K \cdot K = 0$

$\therefore$ $K = \begin{pmatrix} \omega/c \\ \frac{\omega}{c}\cos\theta \\ \frac{\omega}{c}\sin\theta \\ 0 \end{pmatrix}$

in S'



$K' = \begin{pmatrix} \omega'/c \\ \frac{\omega'}{c}\cos\theta' \\ \frac{\omega'}{c}\sin\theta' \\ 0 \end{pmatrix}$

Lorentz transformation $K' = \Lambda K$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\therefore$ $\omega' = \gamma\omega - \gamma\beta\omega\cos\theta$

$\omega'\cos\theta' = -\gamma\beta\omega + \gamma\omega\cos\theta$ $\Rightarrow$ $\cancel{\omega}\frac{\sin\theta}{\sin\theta'} = \gamma\cancel{\omega} - \gamma\beta\cancel{\omega}\cos\theta$

$\omega'\sin\theta' = \omega\sin\theta$

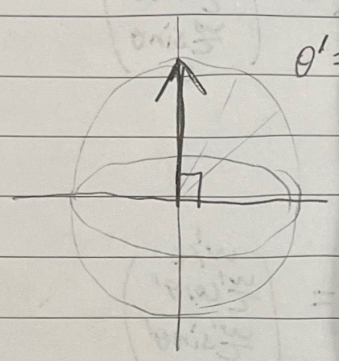
$\therefore$ $\sin\theta' = \dfrac{\sin\theta}{\gamma - \gamma\beta\cos\theta}$

Equivalently

$$-\gamma\beta\omega + \gamma\omega\cos\theta = \gamma\omega\cos\theta' - \gamma\beta\omega\cos\theta\cos\theta'$$

$$\therefore \cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta} \qquad \left(\begin{array}{l}\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\ \beta = \frac{v}{c}\end{array}\right)$$

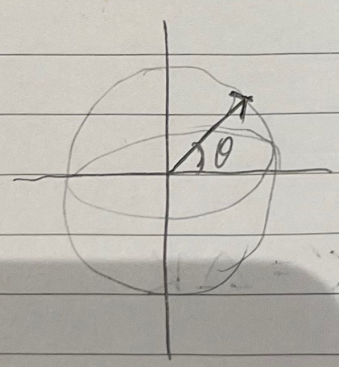
Now, if in rest frame S'



$\theta' = \frac{\pi}{2}$ isotropic emission

Consider the light transverse to the relative motion between S and S' $\left(\theta' = \frac{\pi}{2},\ \cos\theta' = 0\right)$

then in S



$$0 = \frac{\cos\theta - \beta}{1-\beta\cos\theta} \Rightarrow \cos\theta = \beta = \frac{v}{c}$$

$$\therefore \theta = \cos^{-1}\left(\frac{v}{c}\right)$$

- proper acceleration: $U = \begin{pmatrix} \gamma \\ \gamma \underline{v} \end{pmatrix}$ & $A = \frac{dU}{d\tau}$

and in rest frame $\cancel{U = \begin{pmatrix} c \\ 0 \end{pmatrix}}$ $A = \begin{pmatrix} 0 \\ a_0 \end{pmatrix}$

$\left(\because \frac{d\gamma}{d\tau} = \frac{\gamma^3}{c^2}\underline{v}\cdot\dot{\underline{v}}\right.$ and in instantenaous rest frame (IRF)

$\left.\underline{v} = 0\right)$ $\therefore A^0 = 0$ in IRF. $\cancel{\frac{d}{d\tau}(\gamma) = \gamma}$

$\rightarrow$ ~~$U \cdot A = 0$~~

$$\frac{d}{d\tau}(\gamma \underline{v}) = \frac{\gamma^3}{c^2}(\underline{v}\cdot\dot{\underline{v}})\underline{v} + \gamma\dot{\underline{v}}$$ (will be proved in Q2)

$\because$ $\underline{a} = \frac{d\underline{v}}{dt} = \frac{d\tau}{dt}\frac{d\underline{v}}{d\tau} = \frac{1}{\gamma}\frac{d\underline{v}}{d\tau} = \frac{\dot{\underline{v}}}{\gamma}$ $\therefore \dot{\underline{v}} = \gamma\underline{a}$

$$\therefore \frac{d}{d\tau}(\gamma\underline{v}) = \frac{\gamma^4}{c^2}(\underline{v}\cdot\underline{a})\underline{v} + \gamma^2\underline{a}$$

$\therefore$ ~~$A =$~~ $\frac{d}{d\tau}(\gamma c) = c\frac{d\gamma}{d\tau} = \frac{\gamma^3}{c}\underline{v}\cdot\dot{\underline{v}} = \frac{\gamma^4}{c}\underline{v}\cdot\underline{a}$

$$\therefore A = \begin{pmatrix} \frac{\gamma^4}{c}\underline{v}\cdot\underline{a} \\ \frac{\gamma^4}{c^2}(\underline{v}\cdot\underline{a})\underline{v} + \gamma^2\underline{a} \end{pmatrix}$$

If ~~$\forall t$~~ $\underline{v} \perp \underline{a}$, $\underline{v}\cdot\underline{a} = 0$

$\therefore A = \begin{pmatrix} 0 \\ \gamma^2\underline{a} \end{pmatrix}$ $\because$ ~~$U = \begin{pmatrix} \gamma c \\ \gamma\underline{v} \end{pmatrix}$~~

$\because$ $A \cdot A = a_0^2$ $\therefore a_0^2 = (\gamma^2\underline{a})\cdot(\gamma^2\underline{a}) = \gamma^4 a^2$

$\Rightarrow$ $a_0 = \underline{\underline{\gamma^2 a}}$ is proper acceleration

- synchrotron radiation

(i) electrons move in a circular ring

$\rightarrow \underline{v} \perp \underline{a}$

$\therefore a_0 = \gamma^2 a$ we know $E = 3\,GeV = \gamma m_e c^2$

$$\therefore \gamma = \frac{E}{m_e c^2} = \frac{3\times10^3\,MeV}{0.511\,MeV} = 5871$$

$\rightarrow v \simeq c$

$$a = \frac{v^2}{r} \simeq \frac{c^2}{r} = \frac{(3\times10^8)^2}{561} = 1.6\times10^{14}\,m/s^2$$

$$\therefore a_0 = (5871)^2(1.6\times10^{14}) = 5.5\times10^{21}\,m/s^2$$

Radiation power

$$P_L = \frac{\mu_0 q^2 a_0^2}{6\pi c} = \frac{(4\pi\times10^{-7})(1.6\times10^{-19})^2(5.5\times10^{21})^2}{6\pi(3\times10^8)}$$

$$= 1.72\times10^{-10}\,W$$

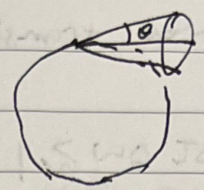
time for one revolution

$$T = \frac{2\pi r}{c} = 1.17\times10^{-5}\,s$$

energy lost per cycle

$$\Delta E = P_L T = 2.02\times10^{-15}\,J = 12.8\,keV$$

(ii) Angular divergence

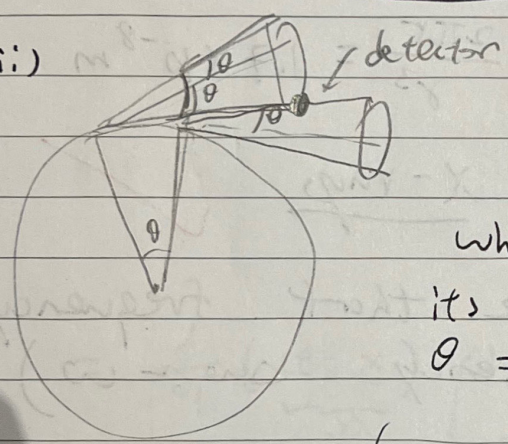


headlight effect $\cos\theta = \frac{v}{c}$ $\therefore \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\gamma}$

$\because \gamma$ is very large $\therefore \theta$ is small $\therefore \sin\theta \simeq \theta$

$\therefore \theta = \frac{1}{\gamma} = \underline{\underline{0.00017 \text{ rad}}}$

(iii)

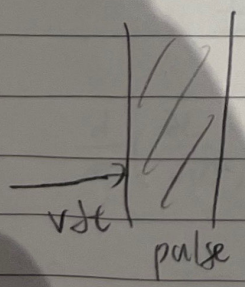


The beam sweeps across the user when the particle moves around its circular orbit by the angle θ = width of lobe $= \frac{1}{\gamma}$ (?)

(or twice that $2\theta = \frac{2}{\gamma}$)

$\therefore$ particle sweeps $r\theta$

time $\delta t = \frac{r\theta}{v} = \frac{r}{\gamma v}$



~~dari~~ duration of pulse is

$$\Delta t = \frac{c\delta t - v\delta t}{c} = \frac{r}{\gamma v}\left(1-\frac{v}{c}\right)$$

$$= \frac{r\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}{\gamma v\left(1+\frac{v}{c}\right)} \underset{v \simeq c}{=} \frac{r}{2c\gamma^3}$$

$$= \underline{\underline{4.6\times10^{-18}\ \text{s}}}$$

$1-\frac{v}{c} \simeq \frac{1}{2\gamma^2}$

(iv) Fourier transform of time gives frequency

$\therefore \quad \Delta t \, \Delta \omega \gtrsim 1$

estimate $\Delta \omega = \frac{1}{\Delta t} = \frac{2c\gamma^3}{r} \propto \underline{\underline{\gamma^3}}$

(v) peak frequency $\omega_c \sim \gamma^3 \omega_0 = \cancel{\gamma^3 \left(\frac{2\pi r}{c}\right)}$

$= \gamma^3 \left(\frac{v}{r}\right) = \gamma^3 \frac{c}{r}$

$\lambda = \frac{2\pi c}{\omega_c} = \frac{2\pi r}{\gamma^3} = 1.7 \times 10^{-8} \, m$

(or $\boxed{87 \, nm}$ using $\omega \sim \Delta\omega = \frac{2c\gamma^3}{r}$)

$\Rightarrow$ <u>X-rays</u>

(we here assume that frequency spread $\sim$ peak frequency $\Delta\omega \sim \omega$)

15 B2 Q2

$$f = f(ct, x, y, z)$$

worldline $\quad (ct(\tau), x(\tau), y(\tau), z(\tau))$

$$\therefore \frac{df}{d\tau} = \frac{\partial f}{\partial (ct)}\frac{d(ct)}{d\tau} + \frac{\partial f}{\partial x}\frac{dx}{d\tau} + \frac{\partial f}{\partial y}\frac{dy}{d\tau} + \frac{\partial f}{\partial z}\frac{dz}{d\tau}$$

$$= \left(\frac{d(ct)}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)\begin{pmatrix} \frac{\partial f}{\partial (ct)} \\ \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

$$= \frac{d}{d\tau}\underbrace{(ct, \underline{x})}_{x^\lambda}\underbrace{\begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \underline{\nabla} \end{pmatrix}}_{\partial_\lambda} f = \frac{dx^\lambda}{d\tau}\partial_\lambda f$$

$$= \underline{\underline{u^\lambda \partial_\lambda f}}$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^a}\right) = \frac{\partial L}{\partial x^a}, \quad L = -mc(-u^\lambda u_\lambda)^{1/2} + q u^\lambda A_\lambda$$

So $$\frac{\partial L}{\partial x^a} = \frac{\partial}{\partial x^a}(q u^\lambda A_\lambda)$$

$$= q u^\lambda \frac{\partial A_\lambda}{\partial x^a} = q u^\lambda \partial_a A_\lambda$$

$$\frac{\partial L}{\partial u^a} = -mc\frac{\partial}{\partial u^a}(-u^\lambda u_\lambda)^{\frac{1}{2}} + q\frac{\partial}{\partial u^a}(u^\lambda A_\lambda)$$

$$\frac{\partial}{\partial u^a}(u^\lambda A_\lambda) = A_\lambda \frac{\partial u^\lambda}{\partial u^a} = A_\lambda \delta^\lambda_a = A_a$$

$$\frac{\partial}{\partial u^a}(-u^\lambda u_\lambda)^{1/2} = -\frac{1}{2}(-u^\lambda u_\lambda)^{-\frac{1}{2}} \frac{\partial}{\partial u^a}(u^\lambda u_\lambda)$$

$$= -\frac{1}{2c}\frac{\partial}{\partial u^a}(g_{\mu\nu}u^\mu u^\nu) = -\frac{1}{2c} g_{\mu\nu}\frac{\partial}{\partial u^a}(u^\mu u^\nu)$$

$u^\lambda u_\lambda = -c^2$

$$= -\frac{1}{2c} g_{\mu\nu}\Big(\underbrace{\frac{\partial u^\mu}{\partial u^a}}_{\delta^\mu_a} u^\nu + \underbrace{\frac{\partial u^\nu}{\partial u^a}}_{\delta^\nu_a} u^\mu\Big)$$

$$= -\frac{1}{2c}\left[g_{\mu\nu}\delta^\mu_a u^\nu + g_{\mu\nu}\delta^\nu_a u^\mu\right]$$

$$= -\frac{1}{2c}\left[\underbrace{g_{\nu a}u^\nu}_{u_a} + \underbrace{g_{\mu a}u^\mu}_{u_a}\right]$$

$$= -\frac{u^a}{c}$$

$$\therefore \frac{\partial L}{\partial u^a} = (-mc)\left(-\frac{u^a}{c}\right) + qA_a$$

$$= mu^a + qA_a$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^a}\right) = u^\lambda \partial_\lambda\left(\frac{\partial L}{\partial u^a}\right) = mu^\lambda \partial_\lambda u^a + qu^\lambda \partial_\lambda A_a$$

$$\therefore m\underbrace{u^\lambda \partial_\lambda u^a}_{\frac{du^a}{d\tau}} = q\underbrace{(\partial_a A_\lambda - \partial_\lambda A_a)}_{F_{a\lambda}}u^\lambda$$

$$\therefore \underline{m\frac{du^a}{d\tau} = qF_{a\lambda}u^\lambda}$$

- The 4-spin :

(i) calculate $u \cdot S = u^\lambda S_\lambda$ in the rest frame of observer $\because u = \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$, $S = \begin{pmatrix} \gamma \underline{S} \cdot \underline{v}/c \\ \gamma \underline{S} \end{pmatrix}$

$\therefore u = \begin{pmatrix} c \\ \underline{0} \end{pmatrix}$ $\quad S = \begin{pmatrix} 0 \\ \underline{S_0} \end{pmatrix}$ $\quad$ ($\underline{S_0}$ = proper 3-spin)

(in rest frame $\underline{v} = \underline{0}$, $\gamma = 1$)

$\therefore u^\lambda S_\lambda = -(c)(0) + \underline{0} \cdot \underline{S} = 0$

this is true in all frames $\Rightarrow$ u and S are orthogonal to each other

(ii) $\dfrac{dS^\alpha}{d\tau} = \dfrac{S_\lambda \dot{u}^\lambda}{c^2} u^\alpha$ $\quad u = \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$ $\quad S = \begin{pmatrix} \gamma \underline{S} \cdot \underline{v}/c \\ \gamma \underline{S} \end{pmatrix}$

$\dot{u} = \begin{pmatrix} \frac{d}{d\tau}(\gamma c) \\ \frac{d}{d\tau}(\gamma \underline{v}) \end{pmatrix}$ $\quad \dfrac{d\gamma}{d\tau} = \dfrac{d}{d\tau}\left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}}$

$\therefore \dfrac{d\gamma}{d\tau} = -\dfrac{1}{2}\left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{3}{2}} \left(-\dfrac{1}{c^2}\right) \dfrac{d}{d\tau}(\underline{v} \cdot \underline{v})$

$= \dfrac{1}{2c^2} \gamma^3 \, 2\underline{v} \cdot \dfrac{d\underline{v}}{d\tau} = \gamma^3 \dfrac{\underline{v} \cdot \dot{\underline{v}}}{c^2}$ $\quad \Rightarrow \dfrac{d(\gamma c)}{d\tau} = \dfrac{\gamma^3}{c} \underline{v} \cdot \dot{\underline{v}}$

$\dfrac{d}{d\tau}(\gamma \underline{v}) = \underline{v} \dfrac{d\gamma}{d\tau} + \gamma \dfrac{d\underline{v}}{d\tau} = \dfrac{\gamma^3}{c^2} (\underline{v} \cdot \dot{\underline{v}}) \underline{v} + \gamma \dot{\underline{v}}$

$S_\lambda \dot{u}^\lambda = -\dfrac{\gamma}{c}(\underline{S} \cdot \underline{v}) \gamma^3 \dfrac{\underline{v} \cdot \dot{\underline{v}}}{c^2} + \left[\dfrac{\gamma^3}{c^2}(\underline{v} \cdot \dot{\underline{v}}) \underline{v} + \gamma \dot{\underline{v}}\right] \cdot \gamma \underline{S}$

$= -\dfrac{\gamma^4}{c^2}(\underline{S} \cdot \underline{v})(\underline{v} \cdot \dot{\underline{v}}) + \dfrac{\gamma^4}{c^2}(\underline{v} \cdot \dot{\underline{v}})(\underline{v} \cdot \underline{S})$

$+ \gamma^2 (\underline{S} \cdot \dot{\underline{v}}) = \gamma^2 (\underline{S} \cdot \dot{\underline{v}})$

$$\therefore \frac{dS^\alpha}{d\tau} = \frac{\gamma^2}{c^2}(\underline{S}\cdot\dot{\underline{V}})\,u^\alpha$$

take the space-components:

$$\therefore \frac{d}{d\tau}(\gamma\underline{S}) = \frac{\gamma^2}{c^2}(\underline{S}\cdot\dot{\underline{V}})(\gamma\underline{V}) = \frac{\gamma^3}{c^2}(\underline{S}\cdot\dot{\underline{V}})\underline{V}$$

$$\frac{d}{d\tau}(\gamma\underline{S}) = \gamma\frac{d\underline{S}}{d\tau} + \underline{S}\frac{d\gamma}{d\tau}$$

$$\therefore \frac{d\underline{S}}{d\tau} = \frac{\gamma^2}{c^2}(\underline{S}\cdot\dot{\underline{V}})\underline{V} - \frac{\underline{S}}{\gamma}\frac{d\gamma}{d\tau}$$

$$= \frac{\gamma^2}{c^2}(\underline{S}\cdot\dot{\underline{V}})\underline{V} - \frac{\underline{S}}{\cancel{\gamma}}\frac{\gamma^{\cancel{3}\,2}}{c^2}(\underline{V}\cdot\dot{\underline{V}})$$

$$\rightarrow \underline{\frac{d\underline{S}}{d\tau} = \frac{\gamma^2}{c^2}\left[(\underline{S}\cdot\dot{\underline{V}})\underline{V} - (\underline{V}\cdot\dot{\underline{V}})\underline{S}\right]}$$

Assume particle is moving along x direction

$\underline{S} = (S_x, S_y, S_z)$, $\underline{V} = (V(\tau), 0, 0)$

$\dot{\underline{V}} = (\dot{V}(\tau), 0, 0)$

$$\therefore \frac{d\underline{S}}{d\tau} = \frac{\gamma^2}{c^2}\left[S_x\dot{V}V\hat{\underline{x}} - \dot{V}V\underline{S}\right]$$

$$\therefore \frac{dS_x}{d\tau} = \frac{\gamma^2}{c^2}\left[S_x\dot{V}V - \dot{V}VS_x\right] = 0 \Rightarrow \underline{S_x(\tau) = S_x(0)}$$

$$\frac{dS_y}{d\tau} = \cancel{\frac{\gamma^2}{c^2}\left[S_x\dot{V}V - \dot{V}V\right.} \frac{\gamma^2}{c^2}\left[-\dot{V}VS_y\right]$$

$$= -\frac{\gamma^2 V\dot{V}}{c^2}S_y$$

$\because \; v(\tau) = c[1 - \exp[-2\Gamma\tau]]^{1/2}$

$$\therefore \dot{v} = \frac{1}{\cancel{2}} c[1 - \exp(-2\Gamma\tau)]^{-\frac{1}{2}} (\cancel{-})(\cancel{2}\Gamma) \exp(-2\Gamma\tau)$$

$$= \frac{c^2\Gamma \exp(-2\Gamma\tau)}{c[1 - \exp(-2\Gamma\tau)]^{\frac{1}{2}}} = \frac{c^2\Gamma \exp(-2\Gamma\tau)}{v}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - (1 - \exp(-2\Gamma\tau))}} = \exp(\Gamma\tau)$$

$$\therefore \quad \cancel{\dot{v} = c^2\Gamma} \qquad \dot{v} = \frac{c^2\Gamma}{v\gamma^2}$$

$$\therefore \quad \gamma^2 v\dot{v} = c^2\Gamma \qquad \therefore \quad \frac{\gamma^2 v\dot{v}}{c^2} = \Gamma$$

$$\therefore \quad \frac{dS_y}{d\tau} = -\Gamma S_y \qquad \therefore \quad \underline{\underline{S_y = S_y(0)\exp(-\Gamma\tau)}}$$

Similarly $\underline{\underline{S_z = S_z(0)\exp(-\Gamma\tau)}}$

- If the particle's acceleration is due to the electro-magnetic force, then

$$\frac{dS^a}{d\tau} = \frac{gq}{2m}\left(F^{a\lambda} S_\lambda + \frac{1}{c^2} F^{\mu\nu} u_\mu S_\nu u^a\right) + \frac{S_\lambda \dot{u}^\lambda}{c^2} u^a$$

magnitude of proper spin is $S_0^2 = \underline{S_0} \cdot \underline{S_0}$

$\because \; S_0^2 = S \cdot S = S^\lambda S_\lambda$

$$\therefore \; \frac{d}{d\tau}(S_0^2) = \frac{d}{d\tau}(S_a S^a) = 2 S_a \frac{dS^a}{d\tau}$$

$$= \frac{gq}{m}\left(\underbrace{F^{a\lambda} S_\lambda S_a}_{②} + \frac{1}{c^2} F^{\mu\nu} u_\mu S_\nu \underbrace{u^a S_a}_{①} \right) + \frac{2 S_\lambda u^\lambda}{c^2} \underbrace{u^a S_a}_{①}$$

① $u^a S_a = 0$ as proved before

② $F^{a\lambda} S_\lambda S_a = 0$ because

$F^{a\lambda} = \partial_a A_\lambda - \partial_\lambda A_a$ is an antisymmetric tensor

$S_\lambda S_a$ is a symmetric tensor

$\Rightarrow$ so $\frac{d}{d\tau}(S_0^2) = 0$ $\therefore$ magnitude of proper spin is constant

15B2Q3

let $c=1$

(a)

~~the~~ $\sum_i (E_i^2 - \underline{p}_i^2 c^2) = \sum_i (E_i^2 - \underline{p}_i^2)$

$= -\sum_i P_i \cdot P_i \qquad \left(P_i = \begin{pmatrix} E_i \\ \underline{p}_i \end{pmatrix} \right)$

$= -\sum_i -m_{0i}^2 = \sum_i m_{0i}^2$

where m_{0i} is the rest mass of the ith particle

$\Rightarrow$ For this to be invariant, we need collisions between these half of the particles to be <u>elastic</u>, with non-changing rest mass.
(pure force)

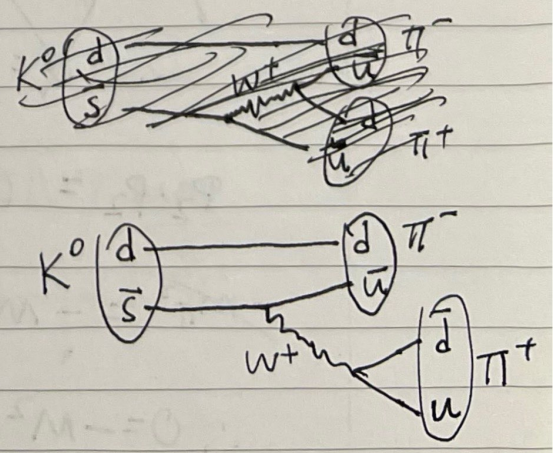
(b) $\left(\sum_i E_i\right)^2 - \left(\sum_i \underline{P}_i\right)^2 = E_{tot}^2 - \underline{P}_{tot}^2 = -P_{tot} \cdot P_{tot}$

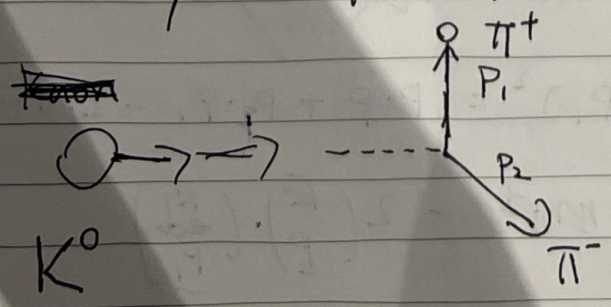
For ~~Pot~~ P_{tot} to be a ~~peper~~ good 4-vector, we need the system to be isolated
(~~Relativity~~ Relativity made relatively easy, pp 81-82)

$\Rightarrow$ And $-P_{tot} \cdot P_{tot}$ is Lorentz invariant if P_{tot} is a good 4-vector

$\Rightarrow$ need ~~&~~ the half of particles to be <u>isolated</u>

- Kaon decay $\quad K^0 \to \pi^+ + \pi^-$

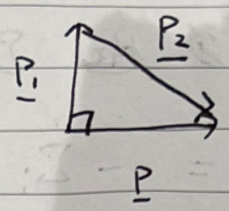




E, M, P

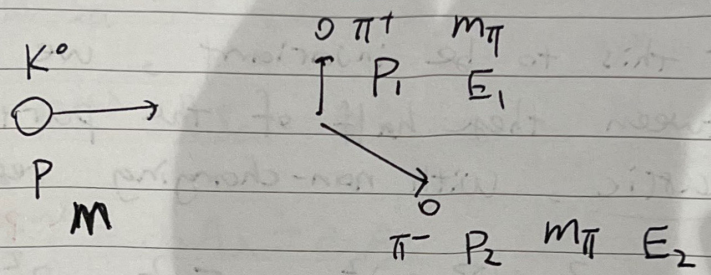
3-momentum conservation

$\underline{P} = \underline{P_1} + \underline{P_2}$



$\because \underline{P_1} \perp \underline{P}$ $\therefore$ we have $P^2 + P_1^2 = P_2^2$

$\rightarrow$ $\underline{P^2 = P_2^2 - P_1^2}$



$P = P_1 + P_2$ $\cancel{P_2 = (P - P_1)}$

$\cancel{\therefore P_2 \cdot P_2 = (P - P_1)^2 = P \cdot P + P_1 \cdot P_1 - 2P \cdot P_1}$

$\cancel{\therefore -m_\pi^2 = -M^2 - m_\pi^2 - 2\begin{pmatrix} E \\ \underline{P} \end{pmatrix} \cdot (}$

$\cancel{P_1 \cdot P_1 = (P - P_2)^2 = P \cdot P + P_2 \cdot P_2 - 2P \cdot P_2}$

$\cancel{\therefore -m_\pi^2 = -M^2 - m_\pi^2 - 2\begin{pmatrix} E \\ \underline{P} \end{pmatrix} \cdot \begin{pmatrix} E_2 \\ \underline{P_2} \end{pmatrix}}$

$\cancel{\therefore 0 = -M^2 - 2(-EE_2 + \underline{P} \cdot \underline{P_2})}$

$P_2 \cdot P_2 = (P - P_1)^2 = P \cdot P + P_1 \cdot P_1 - 2P \cdot P_1$

$\cancel{-m_\pi^2} = -M^2 - \cancel{m_\pi^2} - 2\begin{pmatrix} E \\ \underline{P} \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ \underline{P_1} \end{pmatrix}$

$\therefore 0 = -M^2 - 2(-EE_1 + \underline{P} \cdot \underline{P_1})$ $\because \underline{P} \cdot \underline{P_1} = 0$

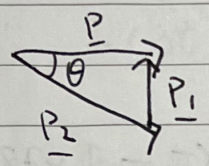
$\therefore M^2 = 2EE_1 \quad \therefore E_1 = \frac{M^2}{2E}$

$\because$ energy conservation $E_1 + E_2 = E$

$\therefore E_2 = E - \frac{M^2}{2E}$

put c back $\Rightarrow$ $\underline{E_2 = E - \frac{M^2c^4}{2E}}$

$= 393.7$ MeV



$\cos\theta = \frac{P}{P_2}$

$P = \sqrt{E^2 - M^2} \qquad P_2 = \sqrt{E_2^2 - m_\pi^2}$

$\therefore \cos\theta = \left(\frac{E^2 - M^2}{\left(E - \frac{M^2c^4}{2E}\right)^2 - m_\pi^2} \right)^{\frac{1}{2}}$

$E = 600$ MeV
$M = 497.6 \frac{MeV}{c^2}$
$m_\pi = 139.6 \frac{MeV}{c^2}$

$= 0.911$

$\therefore \theta = \underline{\underline{24.4^\circ}}$

$\underline{\underline{P_2 = 368.1 \frac{MeV}{c}}}$

$\cancel{P_1 = \sqrt{P_2^2}}$

$E_1 = \frac{M^2}{2E}$

$\therefore P_1 = \sqrt{\left(\frac{M^2}{2E}\right)^2 - m_\pi^2} = \underline{\underline{151.9 \frac{MeV}{c}}}$

$E_2 = 393.7$ MeV

$\because E = \gamma m$

$E_1 = 206.3$ MeV

$\therefore \gamma_1 = \frac{E_1}{m_\pi} = 1.478$

$\gamma_2 = \frac{E_2}{m_\pi} = 2.820$

$\therefore$ in lab frame

- first pion survives $t_1 = \gamma \tau_1 = 1.478 \times 15\,ns$
$= 22.17\,ns$

second pion survives $t_2 = \gamma \tau_2 = 2.820 \times 30\,ns$
$= 84.60\,ns$

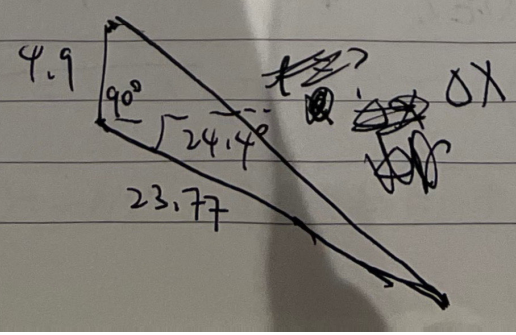
$\because \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ $\therefore 1-\frac{v^2}{c^2} = \frac{1}{\gamma^2}$ $\therefore v = c\sqrt{1-\frac{1}{\gamma^2}}$

$\therefore$ velocities of pions

$v_1 = 2.21 \times 10^8$ m/s , $v_2 = 2.81 \times 10^8$ m/s

$\therefore$ ~~pion~~ distances travelled by each pion

$d_1 = v_1 t_1 = 4.90$ m , $d_2 = v_2 t_2 = 23.77$ m



the distance between events

$$\Delta x = \left(4.9^2 + 23.77^2 - 2 \times 4.9 \times 23.77 \times \cos(90 + 24.4)\right)^{\frac{1}{2}}$$

$$= \underline{26.18\ m}$$

$$\Delta t = t_2 - t_1 = \underline{62.43\ ns}$$

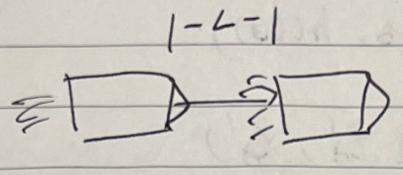
$$\because \underbrace{-c^2\Delta t^2 + \Delta x^2}_{S} = \underbrace{-c^2\Delta t'^2 + \Delta x'^2}_{S'}$$

in frame S', $\Delta t' = 0$ ($\because$ simultaneous)

$$\therefore \Delta x' = \left|\Delta x^2 - c^2\Delta t^2\right|^{\frac{1}{2}} = \underline{\underline{18.29\ m}}$$

~~tbf~~ 15B 2

Q4.



frame S

~~the~~ The string will break

- in frame S. The equilibrium ~~of~~ length of string is contracted to $\frac{L}{\gamma}$, as ~~v~~ rockets accelerates

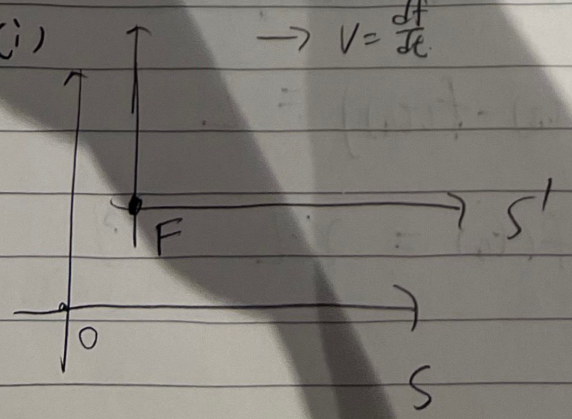
γ gets larger $\frac{L}{\gamma}$ gets smaller, but the length of string remains to be L ∴ more and more ~~str~~ ~~ste~~ stretching → break

- in ~~the~~ the ~~rest~~ instantaneous ~~frest~~ frame S'
~~since the distance between rockets, L, is~~ in which front rocket is at rest, the back rocket is behind much more than L
~~→ consistent → string breaks.~~
Because the distance between rockets, L, is obtained after a Lorentz contraction from the IRF S'. In S' distance $= \gamma L$, string is stretched relative to L ⇒ <u>break</u>

- - - - - - -

(i) → $V = \frac{df}{dt}$

$$\gamma_V = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad V = \frac{df}{dt}$$



In frame S : $A(t_A, f(t_A))$

$B(t_B, h(t_B))$

In frame S' : $A(t_A', x_A')$

$B(t_B', x_B')$

$t_A' = t_B' = t'$

$\cancel{t_A' = \gamma_v(f(t_A) - vt}$

$t_A' = \gamma_v(t_A - \frac{v}{c^2} f(t_A))$ $\quad$ $t_B' = \gamma_v(t_B - \frac{v}{c^2} h(t_B))$

$\because t_A' = t_B'$ $\quad \therefore$ $t_A - \frac{v}{c^2} f(t_A) = t_B - \frac{v}{c^2} h(t_B)$

$\therefore$ $c^2(t_B - t_A) = v(h(t_B) - f(t_A))$ ①

(ii) At time $t_A' = t_B' = t'$ in frame (IRF) S'.

fixed distance $L = x_B' - x_A'$

$x_A' = \gamma_v(f(t_A) - vt_A)$ $\quad$ $x_B' = \gamma_v(h(t_B) - vt_B)$

$\therefore$ $\gamma_v[(h(t_B) - f(t_A)) - v(t_B - t_A)] = L$ ②

(iii) sub ① into ② gives

$\cancel{\gamma_v[\frac{c^2}{v}(t_B - t_A) - v(t_B - t_A)] =}$

$\gamma_v(1 - \frac{v^2}{c^2})(h(t_B) - f(t_A)) = L$

$\sqrt{1 - \frac{v^2}{c^2}}$

$\therefore$ $h(t_B) - f(t_A) = \gamma_v L$ ③

Again sub ① into ② $\quad h(t_B) - f(t_A)$

$$= \frac{c^2}{V}(t_B - t_A)$$

$$\therefore \gamma_V \left[\frac{c^2}{V}(t_B - t_A) - V(t_B - t_A) \right] = L$$

$$\therefore \gamma_V \left[c^2 (t_B - t_A) - V^2 (t_B - t_A) \right] = VL$$

$$\therefore \frac{c^2}{\sqrt{1-\frac{V^2}{c^2}}} \left(1 - \frac{V^2}{c^2}\right)(t_B - t_A) = VL$$

$$\therefore \quad t_B - t_A = \frac{\gamma_V L V}{c^2} \qquad ④$$

let $t_A \equiv t$, $\gamma_V \equiv \gamma$, $\left(V \equiv \frac{df}{dt} \text{ as usual}\right)$

④ $\Rightarrow$ $\quad h(t_B) = h\left(t + \frac{\gamma L V}{c^2}\right)$, $f(t_A) = f(t)$

③ $\Rightarrow$ $\quad \underline{h\left(t + \frac{\gamma L V}{c^2}\right) = f(t) + \gamma L} \qquad ⑤$

— V and γ are both ~~eva~~ evaluated at event A in frame S

If $f(t) = (a^2 + c^2 t^2)^{1/2}$ $\quad V = \frac{df}{dt} = \frac{1}{2}(a^2 + c^2 t^2)^{-\frac{1}{2}} (2c^2 t)$

$$= \frac{c^2 t}{\sqrt{a^2 + c^2 t^2}}$$

$$\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}} = \left[1 - \frac{1}{\cancel{c^2}} \frac{c^{\cancel{4}\,2} t^2}{a^2 + c^2 t^2} \right]^{-\frac{1}{2}}$$

$$= \left[\frac{a^2+c^2t^2}{a^2+c^2t^2} - \frac{c^2t^2}{a^2+c^2t^2} \right]^{-1/2}$$

$$= \left(\frac{a^2}{\left(\sqrt{a^2+c^2t^2}\right)^2} \right)^{-1/2} = \frac{1}{a}\sqrt{a^2+c^2t^2} = \sqrt{1+\left(\frac{ct}{a}\right)^2}$$

$$\gamma v = \frac{1}{a}\cancel{\sqrt{a^2+c^2t^2}} \cdot \frac{c^2t}{\cancel{\sqrt{a^2+c^2t^2}}} = \frac{c^2}{a}t$$

$$t + \frac{\gamma L v}{c^2} = t + \frac{\cancel{c^2}}{a}t\,\frac{1}{\cancel{c^2}}\cdot L = \left(1+\frac{L}{a}\right)t$$

$$\therefore h\left(\left(1+\frac{L}{a}\right)t\right) = f(t) + \sqrt{1+\left(\frac{ct}{a}\right)^2}\,L$$

$$= \sqrt{a^2+c^2t^2} + \frac{L}{a}\sqrt{a^2+c^2t^2}$$

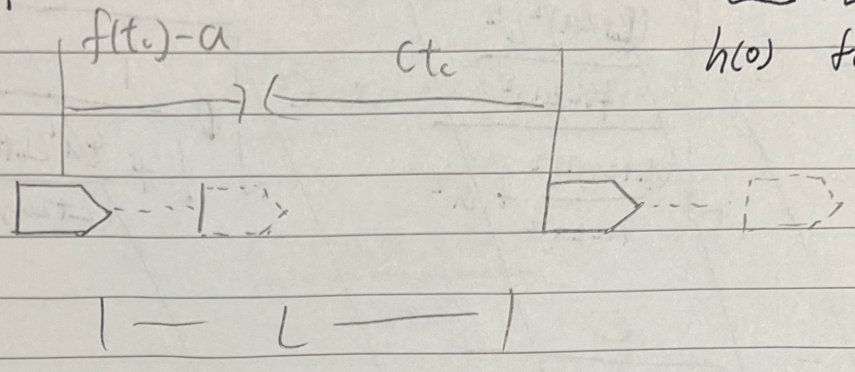
$$= \left(1+\frac{L}{a}\right)\sqrt{a^2+c^2t^2}$$

$$= \left(a^2\left(1+\frac{L}{a}\right)^2 + \left(1+\frac{L}{a}\right)^2c^2t^2\right)^{1/2}$$

$$= \left[(a+L)^2 + c^2\left[\left(1+\frac{L}{a}\right)t\right]^2\right]^{1/2}$$

$$\Rightarrow \quad \underline{\underline{h(t) = \sqrt{(L+a)^2 + c^2t^2}}}$$

$f(t=0) = a$ $\therefore$ F moves by $f(t)-a$ from time $t=0$ to time of arrival of pulse $t=t_c$ in frame S. The distance between ships at time $t=0$ in S is $\underbrace{L+a}_{h(0)} - \underbrace{a}_{f(0)} = \underline{\underline{L}}$



$$\therefore\ f(t_c) - a + ct_c = L$$

$$\therefore\ (a^2 + c^2t_c^2)^{\cancel{1/2}} = [(L+a) - ct_c]^2$$

$$\Rightarrow\ a^2 + \cancel{c^2t_c^2} = (L+a)^2 - \overset{2(L+a)ct_c}{\cancel{2(L+a)}} + \cancel{c^2t_c^2}$$

$$\therefore\ t_c = \frac{(L+a)^2 - a^2}{2(L+a)c} = \underline{\underline{\frac{(L+2a)L}{2(L+a)c}}}$$

$$\cancel{f(t_c) = (a^2 + c^2t_c^2)^{\frac{1}{2}} = \left(a^2 + \frac{(L+2a)^2L^2}{4(L+a)^2}\right)^{1/2}}$$

$$\cancel{= \left(\frac{4a^2(L+a)^2 + (L+2a)L^2}{4(L+a)^2}\right)^{\frac{1}{2}}}$$

$$\cancel{= \left(\frac{4a^2(L^2+2aL+a^2) + (L^2+4aL+4a^2)L^2}{4(L+a)^2}\right)^{\frac{1}{2}}}$$

$$\cancel{= \left(L^4 + 4aL^3 + 8a^2L^2 + 8a^3L + 4a^4\right.}$$

$$V = \frac{c^2 t}{\sqrt{a^2 + c^2 t^2}} = \frac{c^2}{f} t = \frac{c^2}{\left(a^2 + \frac{(L+2a)^2 L^2}{4(L+a)^2}\right)^{1/2}} \left(\frac{(L+2a)L}{2(L+a)c}\right)$$

$$= c \sqrt{\frac{\frac{(L+2a)^2 L^2}{4(L+a)^2}}{\frac{(L+2a)^2 L^2}{4(L+a)^2} + a^2}} = c \sqrt{\frac{1}{1 + \frac{4a^2(L+a)^2}{(L+2a)^2 L^2}}}$$

$$= \frac{c}{\sqrt{1 + \frac{4a^2(L+a)^2}{(L+2a)^2 c^2}}}$$

$$\frac{W_F}{W_H} = \sqrt{\frac{c+V}{c-V}} = \left(\frac{1 + \frac{1}{\sqrt{1 + \frac{4a^2(L+a)^2}{(L+2a)^2 L^2}}}}{1 - \frac{1}{\sqrt{1 + \frac{4a^2(L+a)^2}{(L+2a)^2 c^2}}}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{\sqrt{1 + \frac{4a^2(L+a)^2}{(L+2a)^2 L^2}} + 1}{\sqrt{1 + \frac{4a^2(L+a)^2}{(L+2a)^2 L^2}} - 1}\right)^{\frac{1}{2}}$$

$$\cancel{= \left(1 + \frac{4a^2(L+a)^2}{(L+2a)^2 L^2} + 1 + 2\right.}$$

$$\therefore \quad f(t_c) = L + a - ct_c = L + a - \frac{(L+2a)L}{2(L+a)}$$

$$= \frac{2(L+a)(L+a) - (L+2a)L}{2(L+a)}$$

$$= \frac{2L^2 + 4La + 2a^2 - L^2 - 2aL}{2(L+a)}$$

$$= \frac{L^2 + 2La + 2a^2}{2(L+a)}$$

$$V = \frac{c^2}{f} t$$

$\therefore$ At arrival $\quad V_c = \frac{c^2}{f} t_c = \frac{c\,\cancel{(2(L+a))}}{L^2+2La+2a^2}\left(\frac{\cancel{2}(L+2a)L}{\cancel{2(L+a)}}\right)$

$$\therefore \quad V = \left(\frac{(L+2a)L}{L^2+2La+2a^2}\right)c$$

Doppler effect

$$\frac{\omega_F}{\omega_H} = \sqrt{\frac{c+V}{c-V}} = \left(\frac{1+\frac{V}{c}}{1-\frac{V}{c}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{L^2+2La+2a^2+L^2+2aL}{L^2+2La+2a^2-L^2-2aL}\right)^{\frac{1}{2}}$$

$$= \left(\frac{\cancel{2}(a+L)^2}{\cancel{2}a^2}\right)^{\frac{1}{2}} = \frac{a+L}{a}$$