2702

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B2. SYMMETRY AND RELATIVITY

TRINITY TERM 2014

Wednesday, 18 June, 2.30 pm – 4.30 pm

10 minutes reading time

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

$-P_2$ $\quad P_2$ $\quad M^2 - 2P_2M + P_2^2 = P_2^2 + m^2$

$(P_2^2 + m^2)^{1/2} + P_2 = M$ $\quad \therefore M^2 - m^2 = 2P_2M$

$(M - P_2)^2 = P_2^2 + m^2$ $\quad P_2 = \frac{M^2 - m^2}{2M}$

1. An atom has rest mass m when in its ground state and rest mass M when in an excited state. Such an atom is at rest in the laboratory, in the excited state, and then emits a photon as it decays to the ground state. Find, in terms of m and M, an expression for the energy of the photon. [2]

The neutral calcium atom has a very narrow transition line at 729 nm. Two calcium atoms are moving directly towards one another, one in the excited state associated with this transition, the other in the ground state. Show that, if a photon emitted by the first atom is absorbed by the second, then the relative speed v_r of the atoms is the same before and after the process (only the direction of relative motion changes). Find an expression for v_r in terms of m and M and give its value for this transition in calcium. (The relative atomic mass of calcium is 40). [7]

Find expressions, in terms of m and M, for the total energy in the CM frame and the speed of the CM frame relative to the initial rest frame of one of the atoms. Hence, or otherwise, show that the initial momentum of either atom in the CM frame is

$$p_{\rm cm} = \frac{M^2 - m^2}{\sqrt{8(M^2 + m^2)}}.$$

[8]

Consider the same process (that is, a photon emitted by one atom is absorbed by another atom of the same type), but now for a nuclear transition in which $M = \sqrt{5}m$. Describe the initial and final conditions in the CM frame, first in terms of momentum and then in terms of velocity. Draw an accurate spacetime diagram showing the worldlines of the atoms and the exchanged photon, paying attention to the slopes of the worldlines before and after the process. [8]

$M^2 = 5m^2$

$5 + \frac{1}{3} = \frac{16}{3}$ $\quad \left(\frac{4}{\sqrt{13}}\right)$ $\quad \frac{6}{13}$ $\quad v_r$ $\quad M$ $\quad m$

P_γ $\quad m$ $\quad M$ $\quad v_r$

$\frac{M^2 + m^2}{2M} - \frac{2M^2 - 2Mm}{2M}$

$= \frac{-1}{2M}[M^2 - 2Mm + m^2]$

$= -\frac{1}{2M}(M - m)^2$

$m(v_r) +$

$0 + 2P_\gamma m + m^2 = M^2$

$\therefore P_\gamma = \frac{M_1^2 - m^2}{2m} = \frac{M_2^2 + m^2}{2m_2}$

$M^2 = E_1 P_2$

$\therefore$ ~~$M_1^2 - m$~~

$E_1 \quad E_2$

$P_1 \quad P_2$

$2M_1^2 M_2 - 2m_2 m^2 = 2M_2^2 m + 2m^3$

2. Inertial frames S and S′ are in standard configuration; that is, their axes are aligned and S′ moves relative to S in the x direction at speed v. A particle moves with speed u at angle θ to the x-axis of frame S. Show that the angle between the velocity vector of this particle and the x' axis of frame S′ is given by

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - v/u)}$$

and give the definition of the factor γ in this result. [4]

A photon of energy 10 MeV is incident on an electron at rest in the laboratory and undergoes elastic scattering. If the photon emerges at an angle of 25° to its initial direction in the laboratory frame, find the scattering angle in the CM frame, and the angle at which the electron emerges in the laboratory frame. [9]

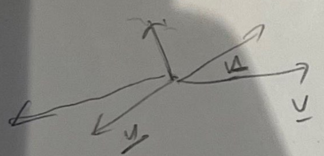
Show that, if two particles have velocities $\mathbf{u}$, $\mathbf{v}$ relative to some frame, then the speed of one particle relative to the other is

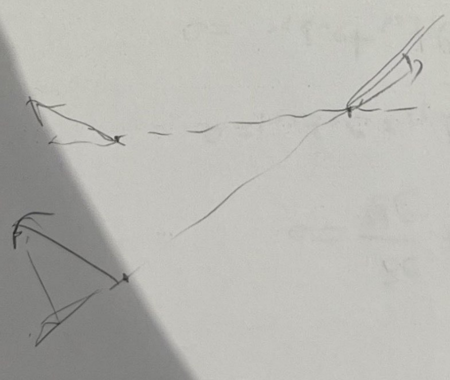
$$w = \frac{\sqrt{(\mathbf{u}-\mathbf{v})^2 - (\mathbf{u}\wedge\mathbf{v})^2/c^2}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}$$

[Hint: you may find it useful to show that, for any vectors $\mathbf{a}$, $\mathbf{b}$, $(\mathbf{a}\wedge\mathbf{b})^2 = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2$.] [8]

A long spring is extended by a large amount and then released. In what inertial frame is the rate of change of the length of the spring greatest? What is the maximum rate at which this length can change? (A thorough answer can be given without entering into lengthy algebraic analysis). [4]

$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$ $\quad u_y' = \frac{u_y}{\gamma(1 - \frac{u_x v}{c^2})}$





$J^b = \rho_0 U^b = \begin{pmatrix} \rho c \\ \underline{u} \end{pmatrix}$

3. Write down the relationship between electric and magnetic fields and the scalar and vector potentials. The Faraday tensor is related to the four-vector potential by the expression

$$F^{ab} = \partial^a A^b - \partial^b A^a.$$

Use this to find the expression for F^{ab} in terms of electric and magnetic fields $\mathbf{E}$, $\mathbf{B}$. [3]

The electromagnetic field satisfies $\partial_\lambda F^{\lambda b} = -\mu_0 J^b$ where J^b is the four-current. Use this equation to obtain two of the Maxwell equations. Also, obtain the other two Maxwell equations given that F^{ab} can be obtained from a 4-potential. [6]

The stress-energy tensor of the electromagnetic field may be written

$$T^{ab} = \epsilon_0 c^2 \left(-F^{a\lambda} F^b_\lambda - \tfrac{1}{4} g^{ab} F_{\mu\nu} F^{\mu\nu} \right)$$ $F^{\mu\lambda}F^{b}_{\lambda}$ $\frac{1}{2}(F^{a\lambda}F^{b}_{\lambda} + F^{a}_{\lambda}F^{b\lambda})$

$= F^{b\lambda}F^{a}_{\lambda}$

Find the top row of this tensor (i.e. T^{0b}) in terms of $\mathbf{E}$ and $\mathbf{B}$ for a general field, and identify the physical quantities obtained. Find the $b = 0$ component of $\partial_\lambda T^{\lambda b}$ in terms of $\mathbf{E}$ and $\mathbf{B}$, and state how it is related to the current density $\mathbf{j}$. [5]

A parallel-plate capacitor has its plates parallel to the xz plane and moves relative to the laboratory in the x direction at speed v. Let E be the electric field between the plates in the rest frame of the capacitor. Write down the Faraday tensor for this field in the rest frame, and hence obtain the stress-energy tensor, first in the rest frame and then in the laboratory. Hence find the Poynting vector of the field observed in the laboratory. [5]

Repeat the calculation for a capacitor moving in the same way whose plates are parallel to the yz plane. In both cases describe the physical processes whereby the capacitor's stored energy is transported. [6]

$F^{ab} = \partial^a A^b - \partial^b A^a$

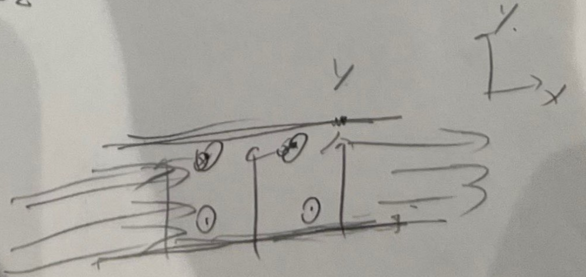
$\partial^c F^{ab} + \partial^b F^{ca} + \partial^a F^{bc} = 0$

$\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0$

$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$

$$\left(\begin{array}{c|ccc} 0 & & E/c & \\ \hline & 0 & B_z & -B_y \\ -E/c & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{array}\right)$$

$\partial^0 F^{13} + \partial$



4. Write down the scalar and vector potential for the field of a charged particle at rest. Hence, carefully explaining your reasoning, show that the 4-vector potential of an arbitrarily moving charged particle is given by

$$\mathsf{A} = \frac{q}{4\pi\epsilon_0} \frac{\mathsf{U}/c}{(-\mathsf{R}\cdot\mathsf{U})}$$

and define the quantities U and R involved in this expression. Make sure you justify the claim that your argument leads to a genuinely covariant result. [5]

A particle of charge q is moving at constant speed around a circle in the xy plane, such that its position is given by $(x, y, z) = (a\cos\omega t_s,\ a\sin\omega t_s,\ 0)$ at any given time t_s. It is desired to obtain the electric field $\mathbf{E}$ at points on the z axis. To this end, find the source time t_s for a field event occurring at $(0, 0, z)$ at time t. Let $\mathbf{v}_s$ be the velocity of the particle at the source time. Find an expression for A in terms of $q, a, z, \mathbf{v}_s$ and fundamental constants. Hence obtain E_z. [10]

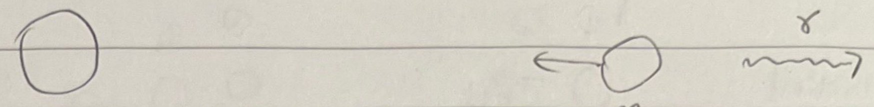
To find the other components of $\mathbf{E}$, the gradient of the scalar potential ϕ is required. Consider a field event at $(\Delta x,\ 0,\ z)$ at time t, and obtain the dependence of ϕ on Δx, to first order. Hence show that the x-component of the electric field at $(0, 0, z)$ is given by

$$E_x = \frac{qa}{4\pi\epsilon_0 c^2\sqrt{a^2+z^2}}\left(\left(\omega^2 - \frac{c^2}{a^2+z^2}\right)\cos\omega t_s + \frac{\omega c}{\sqrt{a^2+z^2}}\sin\omega t_s\right)$$

[10]

14B2Q1 $\qquad c = 1$

Lab, before $\qquad\qquad$ Lab after



$M \qquad m \qquad \gamma$

$$P_M = \begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad P_m = \begin{pmatrix} E_1 \\ P_1 \\ 0 \\ 0 \end{pmatrix} \qquad P_\gamma = \begin{pmatrix} P_2 \\ P_2 \\ 0 \\ 0 \end{pmatrix}$$

~~Conservation of energy.~~

$$\cancel{M = E_1 + P_2}$$

~~Conservation of~~ momen

$$P_M = P_m + P_\gamma \qquad \therefore \; P_\gamma = P_M - P_m$$

$$\therefore \; P_\gamma^2 = P_M^2 + P_m^2 - 2P_M \cdot P_m$$

$$0 = -M^2 - m^2 - 2[-ME_1]$$

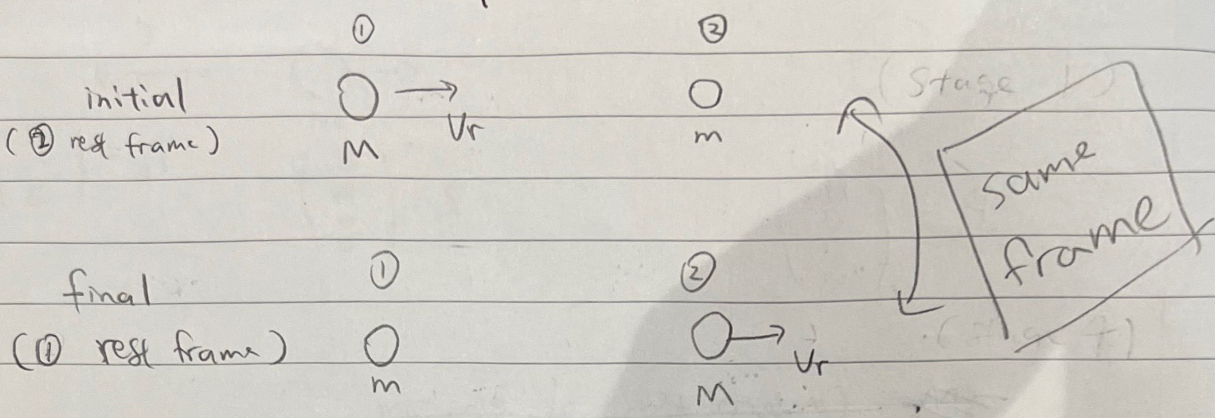
$$\therefore \; 2ME_1 = M^2 + m^2$$

$$\therefore \; E_1 = \frac{M^2 + m^2}{2M} = \text{energy of atom (ground state)}$$

$\therefore$ energy of photon $\cancel{E_\gamma = E_1 = \frac{M^2+m^2}{2M}}$

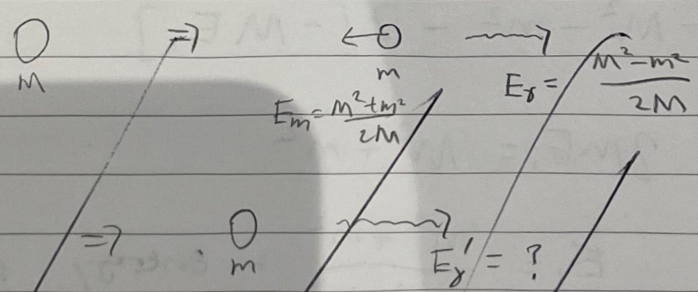
$$E_\gamma = M - E_1 = \boxed{\frac{M^2 - m^2}{2M}}$$

- The emission absorption process :

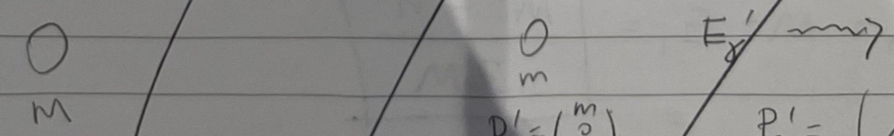


Since the masses exchange after the emission -absorption process, by symmetry, the relative ~~the~~ speed must remain the same. to conserve both energy and momentum.

- $(M - m = \frac{hc}{729nm})$



- we find the energy of photon in the rest frame of particle ① after ~~its mass has changed~~ it emitted the photon



$P_M = \begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $P_1' = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $P_2' = \begin{pmatrix} E_\gamma' \\ E_\gamma' \\ 0 \\ 0 \end{pmatrix}$

$$P_M^2 = (P_1'^2 + P_2')^2$$

look at it in another way: (initial rest frame of ①)

initial: ① $\bigcirc$ M, $v=0$ ② $\leftarrow\bigcirc$ V_r m

$E_m = \gamma_r m$ $\quad E_\gamma = \frac{M^2-m^2}{2M}$ $\quad \gamma_r = (1-V_r^2)^{-\frac{1}{2}}$

intermediate: $\leftarrow\bigcirc$ V_r m $\rightsquigarrow \gamma$ $\leftarrow\bigcirc$ V_r m

final: $\leftarrow\bigcirc$ V_r m $\quad\bigcirc$ M, $V=0$

$$\therefore \gamma_r m + \frac{M^2-m^2}{2M} = M$$

$$\gamma_r m = \frac{M^2+m^2}{2M} \qquad \therefore \gamma_r = \frac{M^2+m^2}{2Mm}$$

$$\because \gamma_r = (1-V_r^2)^{-\frac{1}{2}} \qquad \therefore \cancel{V_r =} \ 1-V_r^2 = \frac{1}{\gamma_r^2}$$

$$\therefore V_r = \sqrt{1-\frac{1}{\gamma_r^2}} = \sqrt{1-\frac{4M^2m^2}{(M^2+m^2)^2}}$$

$$= \sqrt{\frac{M^4+2M^2m^2+m^4-4M^2m^2}{M^2+m^2}} = \sqrt{\frac{(M^2-m^2)^2}{(M^2+m^2)^2}} = \boxed{\frac{M^2-m^2}{M^2+m^2}c}$$

$$\cancel{V_r = \frac{M^2-m^2}{M^2+m^2} = \frac{M^2-m^2}{2M} \cdot \frac{2M}{M^2+m^2}}$$

$$\cancel{\therefore E_\gamma = \frac{M^2+m^2}{2M}} \qquad c^2(M-m) = \frac{hc}{\lambda}$$

$$\therefore M-m = \frac{h}{\lambda c} = \frac{6.63\times10^{-34}}{729\times10^{-9}\times3\times10^{8}} = 3\times10^{-36}\,kg$$

$$m = 40 \times 1.66\times10^{-27}\ \text{kg}$$

$$= 6.64\times10^{-26}$$

$$\therefore \quad M^2 \simeq m^2$$

$$V_r = c\,\frac{(M+m)(M-m)}{M^2+m^2} \simeq c\,\frac{\cancel{2m}(M-m)}{\cancel{2}m^{\cancel{2}}}$$

$$\simeq c\,\frac{M-m}{m} = \frac{3\times10^{-36}}{6.64\times10^{-26}}\,(3\times10^{8})$$

$$= \boxed{0.0136\ \text{m/s}}$$

— cm frame

$E_1 = \frac{M^2+m^2}{2M}$ $\quad E_2 = M$ $\quad \gamma_r = \frac{M^2+m^2}{2Mm}$

$m \rightarrow V_r$ $\quad M$ $\quad V_r = \frac{M^2-m^2}{M^2+m^2}$

$P_1 = V_r E_1$ $\quad P_2 = 0$

total energy of cm frame is

$$E_{cm}^2 = E_{lab}^2 - P_{lab}^2$$

$$= (E_1+E_2)^2 - (P_1+P_2)^2$$

$$= \left(\frac{M^2+m^2}{2M} + M\right)^2 - \left(\frac{M^2-m^2}{\cancel{M^2+m^2}}\,\frac{\cancel{M^2+m^2}}{2M}\right)^2$$

$$= \left(\frac{M^2+m^2}{2M} + M\right)^2 - \left(\frac{M^2-m^2}{2M}\right)^2$$

$$= \frac{1}{4M^2}\left[(M^2+m^2)^2 + 4(M^2+m^2)M^2 + 4M^4 - (M^2-m^2)^2\right]$$

$$= \frac{1}{4M^2}\left[(2M^2)(2m^2) + 4(M^2+m^2)M^2 + 4M^4\right]$$

$$= \frac{1}{4M^2}\left[4M^2m^2 + 4M^4 + 4m^2M^2 + 4M^4\right]$$

$$= 2(M^2+m^2) \Rightarrow \boxed{\sqrt{2(M^2+m^2)} = E_{cm}}$$

- CM velocity

$$V_{cm} = \frac{P_{tot}}{E_{tot}} = \frac{P_{lab}}{E_{lab}}$$

$$= \frac{\frac{M^2-m^2}{2M}}{\frac{M^2+m^2}{2M} + M} = \frac{M^2-m^2}{M^2+m^2+2M^2} = \frac{M^2-m^2}{3M^2+m^2}$$

- In CM frame momentum of M, since M has velocity $= V_{cm}$ (at rest at lab frame), is:

$$P_{cm} = \frac{M V_{cm}}{\sqrt{1-V_{cm}^2}} = \left(\frac{M^2-m^2}{3M^2+m^2}\right)M \Bigg/ \sqrt{\frac{9M^4+6m^2M^2+\cancel{m^4}-M^4+2m^2M^2-\cancel{m^4}}{(3M^2+m^2)^2}}$$

$$= \left(\frac{M^2-m^2}{\cancel{3M^2+m^2}}\right)M \Bigg/ \sqrt{\frac{8M^4+8m^2M^2}{\cancel{(3M^2+m^2)^2}}}$$

$$= \frac{(M^2-m^2)\cancel{M}}{\cancel{M}\sqrt{8(M^2+m^2)}} = \boxed{\frac{M^2-m^2}{\sqrt{8(M^2+m^2)}}}$$

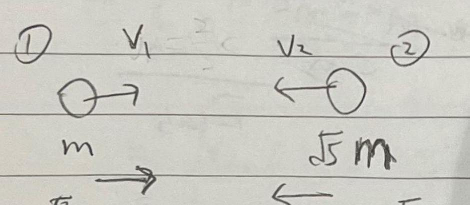
- nuclear transition $M = \sqrt{5}\, m$ ∴ $M^2 = 5m^2$

$$V_r = \frac{2}{3}c \qquad \gamma_r = \frac{3\sqrt{5}}{5} = \frac{3}{\sqrt{5}} \simeq 1.34$$

$$P_{cm} = \frac{4}{\sqrt{8\times 6}}\, m = \frac{1}{\sqrt{3}}\, m = \frac{\sqrt{3}}{3}\, m$$

$$E_{cm} = 2\cancel{(\sqrt{5}m + m) = 12m} \quad \sqrt{2(6)} = 2\sqrt{3}\, m$$

initial initial



$P_1 = \frac{\sqrt{3}}{3} m$ $\qquad P_2 = \frac{\sqrt{3}}{3} m$

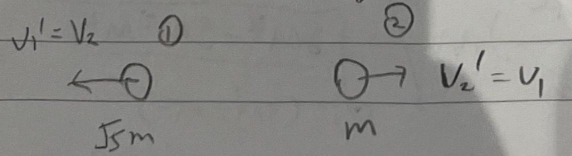
$E_1 = \left(\frac{1}{3} + 1\right)^{\frac{1}{2}} m \qquad E_2 = \left(\frac{1}{3} + 5\right)^{1/2} m$

$= \frac{2}{\sqrt{3}} m \qquad = \frac{4}{\sqrt{3}} m$

$\gamma_1 = \frac{2}{\sqrt{3}}$ ∴ $V_1 = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}c \simeq 0.5c$

$\gamma_2 = \frac{4}{\sqrt{3}\cdot\sqrt{5}}$ ∴ $V_1 = \sqrt{1 - \frac{15}{16}} = \frac{c}{4} \simeq 0.25c$

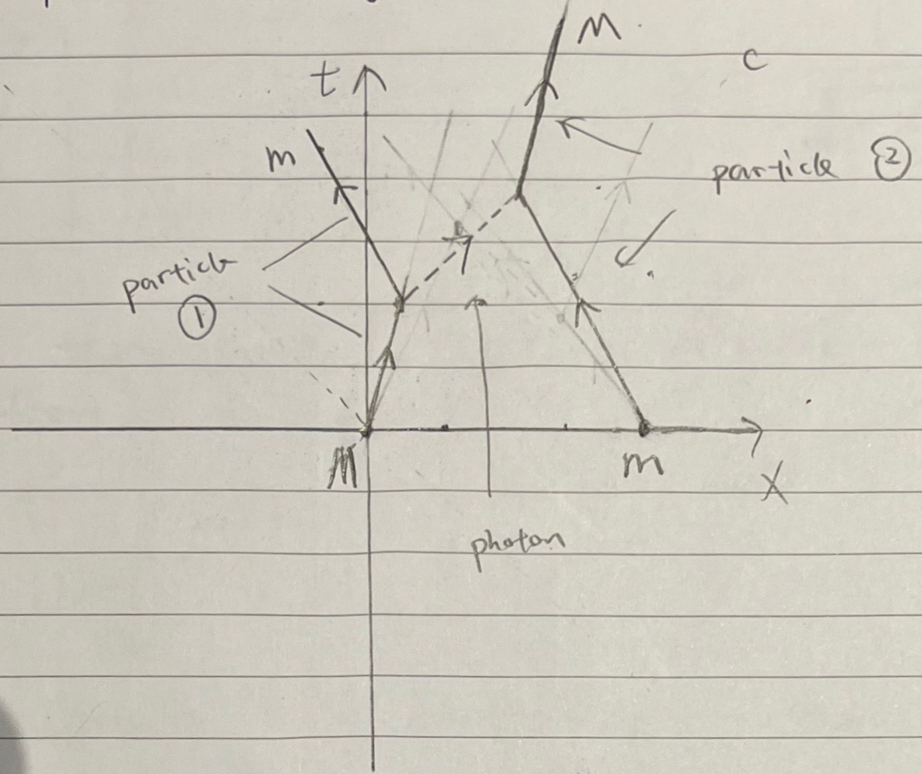
final (exchange mass & velocity, momentum)



$P_2' = P_1 = \frac{\sqrt{3}}{3} m \qquad V_1' = V_2 = \frac{c}{4} \simeq 0.25$

$P_1' = P_2 = \frac{\sqrt{3}}{3} m \qquad V_2' = V_1 = \frac{1}{2}c \simeq 0.5c$

- spacetime diagram : CM frame



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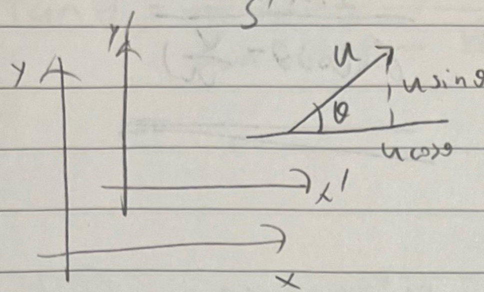
$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$



~~transformation of 4-velocity~~ $\cancel{U = \begin{pmatrix} \gamma c \\ \gamma \underline{u} \end{pmatrix}}$

~~from S ⇒ S' is~~ $\cancel{U' = \Lambda U}$

$\cancel{U' = [}$

Lorentz transformation : from S to S'

$$x' = \gamma(x - vt) \quad y' = y \quad z' = z \quad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

~~speed at~~ velocity components at S' : (u_x', u_y', u_z')

at S : (u_x, u_y, u_z)

$\cancel{u_x' = dx'}$

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right) = \gamma\left(1 - \frac{vu_x}{c^2}\right)$$

$$u_x' = \frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \frac{\gamma(u_x - v)}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy'}{dt}\frac{dt}{dt'} = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

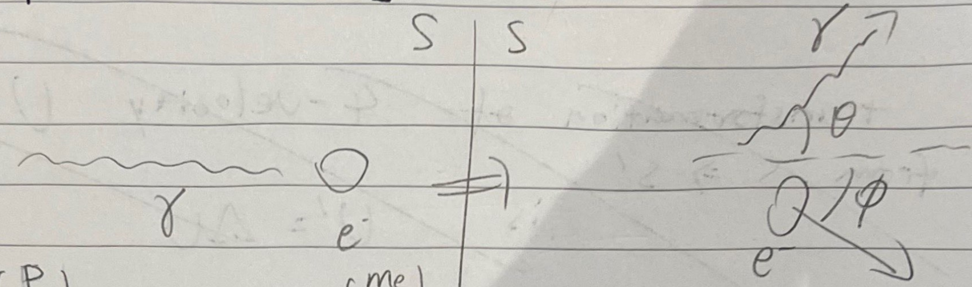
$u_x = u\cos\theta$, $u_y = u\sin\theta$

$$\tan\theta' = \frac{u_y'}{u_x'} = \frac{u\sin\theta / \gamma\left(1 - \frac{vu\cos\theta}{c^2}\right)}{(u\cos\theta - v) / \left(1 - \frac{vu\cos\theta}{c^2}\right)}$$

$$= \frac{u\sin\theta}{\gamma(u\cos\theta - v)} = \frac{\sin\theta}{\gamma(\cos\theta - \frac{v}{u})}$$

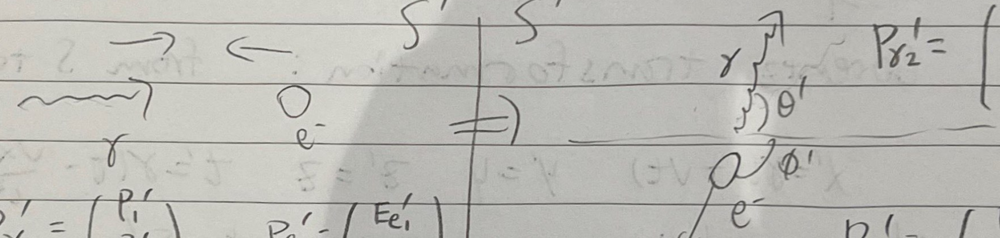
– compton scattering

$c = \hbar = 1$



$P_{\gamma} = \begin{pmatrix} P_2 \\ \underline{P_2} \end{pmatrix}$

$P_{\gamma_1} = \begin{pmatrix} P_1 \\ P_1 \\ 0 \\ 0 \end{pmatrix}$ $\quad P_{e_1} = \begin{pmatrix} m_e \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad P_{e_2} = \begin{pmatrix} E_{e2} \\ \underline{P_{e2}} \end{pmatrix}$



$P'_{\gamma_2} = \begin{pmatrix} P'_2 \\ \underline{P'_2} \end{pmatrix}$

$P'_{\gamma_1} = \begin{pmatrix} P'_1 \\ P'_1 \\ 0 \\ 0 \end{pmatrix}$ $\quad P'_{e_1} = \begin{pmatrix} E'_{e_1} \\ P'_{e_1} \\ 0 \\ 0 \end{pmatrix}$ $\quad P'_{e_2} = \begin{pmatrix} E'_2 \\ \underline{P'_{e_2}} \end{pmatrix}$

$\theta = 25^\circ$ in S frame

photon: $P_1 = E_1 = 10\,\text{MeV}$

electron: $m_e = 0.511\,\text{MeV}$, $P_{e_1} = 0$

velocity of CM frame (S') relative to S is

$$\beta_{cm} = \frac{P_{tot}}{E_{tot}} = \cancel{\frac{10}{10+}} \frac{10}{10+0.511} = 0.9514$$

$$\therefore \gamma_{cm} = \frac{1}{\sqrt{1-(0.9514)^2}} = 3.25$$

Angle transformation $\cancel{\theta}$

$$\tan\theta' = \frac{\sin\theta}{\gamma_{cm}(\cos\theta - \frac{V_{cm}}{c})} = \frac{\sin\theta}{\gamma_{cm}(\cos\theta - \beta_{cm})}$$

$u = c =$ speed of photon.
$= 1$

$$= \frac{\sin(25^\circ)}{3.25(\cos 25^\circ - 0.9514)} = -2.884$$

$$\therefore \theta' = \underline{109.12^\circ}$$

in cm frame, electron angle $\phi' = 180^\circ - \theta'$

$$\therefore \phi' = 70.88^\circ$$

use angle formula from S' to S, then

$V = -V_{cm}$, $u =$ electron velocity in cm frame
after collision

to find $u = V_{e_2}'$:

$$E_{cm} = \sqrt{E_{tot}^2 - P_{tot}^2} = \sqrt{10.511^2 - 10^2} = 3.237 \text{ MeV}$$

$$P_2' + E_{e_2}' = E_{cm}$$

$$P_2'^2 = P_{e_2}'^2 \qquad (\underline{P_{e_2}'} = -\underline{P_2'}) \Leftarrow \text{cm frame}$$

$$\therefore P_{e_2}' + \sqrt{P_{e_2}'^2 + m_e^2} = E_{cm}$$

$$\therefore \cancel{P_{e_2}'^2} + m_e^2 = E_{cm}^2 - 2E_{cm}P_{e_2}' + \cancel{P_{e_2}'^2}$$

$$\therefore P_{e_2}' = \frac{E_{cm}^2 - m_e^2}{2E_{cm}} = \frac{3.237^2 - 0.511^2}{2\times 3.237} = 1.578 \text{ MeV}$$

$$E'_{e_2} = \sqrt{P'^2_{e_i} + m_e^2} = 1.659 \text{ MeV}$$

$$\therefore \gamma_u = \frac{1.659 \text{ MeV}}{m_e c^2} = \frac{1.659}{0.511} = \underline{3.247}$$

$u = 0.9514$ (electron velocity $= V_{cm}$ $\because$ ~~etc~~ elastic scattering + ~~trans~~ boost from rest)

$$\therefore \tan\phi = \frac{\sin\phi'}{\gamma_{cm}(\cancel{\sin}\cos\phi' + \frac{V_{cm}}{u})}$$

$$= \frac{\sin(70.88)}{3.25(\cos(70.88)+1)}$$

$$= 0.219$$

$$\underline{\phi = 12.35^\circ}$$

— $\underline{a} \times \underline{b} = \varepsilon_{ijk} a_j b_k$

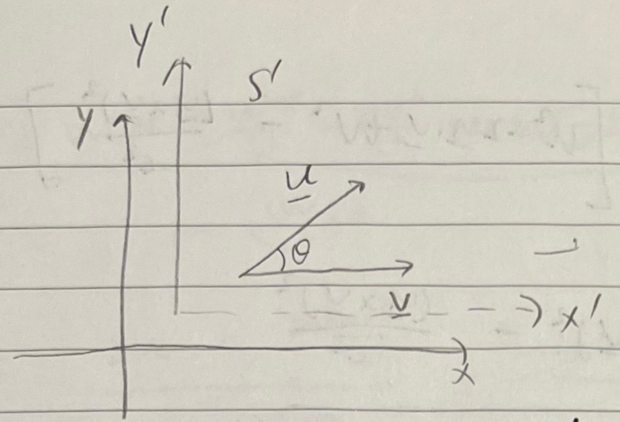
$$\therefore (\underline{a} \times \underline{b})^2 = \varepsilon_{ijk} a_j b_k \varepsilon_{ilm} a_l b_m$$

$$= \varepsilon_{ijk}\varepsilon_{ilm} a_j b_k a_l b_m$$

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) a_j b_k a_l b_m$$

$$= a_j a_j b_k b_k - a_j b_j a_k b_k$$

$$= \underline{a^2 b^2 - (\underline{a} \cdot \underline{b})^2}$$



$u_x = u\cos\theta$

$u_y = u\sin\theta$

$\underline{u} = (u_x, u_y, 0)^T$

$\underline{u}\cdot\underline{V} = u_x V \Leftarrow \underline{V} = (V, 0, 0)$

Orient the x coordinate of frame S along $\underline{V}$

frame S' moving at velocity $\underline{V}$ relative to S

transform to S' from S:

$$u_x' = \frac{u_x - V}{1 - \frac{u_x V}{c^2}}, \quad u_y' = \frac{u_y}{\gamma\left(1 - \frac{u_x V}{c^2}\right)}$$

$$W^2 = u_x'^2 + u_y'^2 = \frac{(u_x - V)^2 + (u_y/\gamma)^2}{\left(1 - \frac{u_x V}{c^2}\right)^2}$$

$$= \frac{1}{\left(1 - \frac{u_x V}{c^2}\right)^2}\left[u_x^2 - 2u_x V + V^2 + \frac{u_y^2}{\gamma^2}\right].$$

$$= \frac{1}{\left(1 - \frac{u_x V}{c^2}\right)^2}\left[u_x^2 - 2u_x V + V^2 + \left(1 - \frac{V^2}{c^2}\right)u_y^2\right]$$

$$= \frac{1}{1 - \frac{u_x V}{c^2}}\left[(u_x^2 + u_y^2) - 2u_x V + V^2 - \frac{V^2 u_y^2}{c^2}\right]$$

$$\underline{u}\times\underline{V} = \begin{vmatrix} i & j & k \\ u_x & u_y & 0 \\ V & 0 & 0 \end{vmatrix} = -Vu_y\underline{\hat{k}} \quad \therefore (\underline{u}\times\underline{V})^2 = u_y^2 V^2$$

$\because \underline{u}\cdot\underline{V} = u_x V$

and $u^2 = u_x^2 + u_y^2$

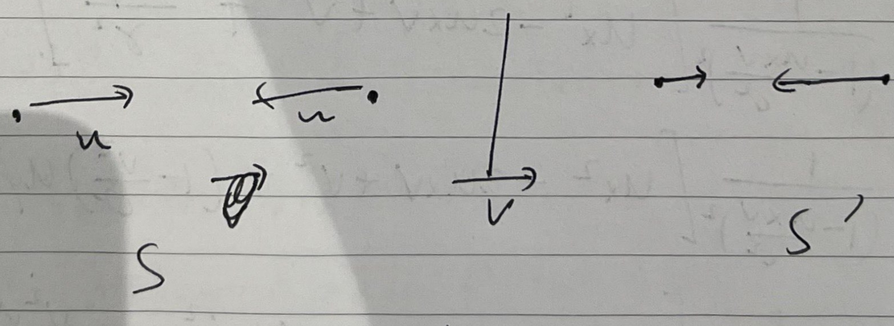
$$\therefore\ w^2 = \frac{1}{\left(1-\frac{\underline{u}\cdot\underline{v}}{c^2}\right)}\left[u^2 - 2\underline{u}\cdot\underline{v} + v^2 - \frac{(\underline{u}\times\underline{v})^2}{c^2}\right]$$

$$= \frac{(\underline{u}-\underline{v})^2 - \frac{(\underline{u}\times\underline{v})^2}{c^2}}{\left(1-\frac{\underline{u}\cdot\underline{v}}{c^2}\right)^2}$$

$$\therefore\ w = \frac{\sqrt{(\underline{u}-\underline{v})^2 - (\underline{u}\times\underline{v})^2/c^2}}{1-\frac{\underline{u}\cdot\underline{v}}{c^2}}$$

- closing speed

consider a frame S, in which two ends of the spring has same speed u (opposite directions), so ~~len~~ rate of change of length $\dot{l} = 2u$



consider a ~~para~~ longitudinal boost.

$$\text{Now } \dot{l}' = \frac{u-v}{1-\frac{uv}{c^2}} + \frac{u+v}{1+\frac{uv}{c^2}}$$

$$= \frac{\left(1+\frac{uv}{c^2}\right)(u-v) + \left(1-\frac{uv}{c^2}\right)(u+v)}{1-\left(\frac{uv}{c^2}\right)^2}$$

$$= \frac{u - \cancel{v} + \cancel{\frac{u^2 v}{c^2}} - \frac{uv^3}{c^2} + u + \cancel{v} - \cancel{\frac{u^2 v}{c^2}} - \frac{uv^2}{c^2}}{1 - \left(\frac{uv}{c^2}\right)^4}$$

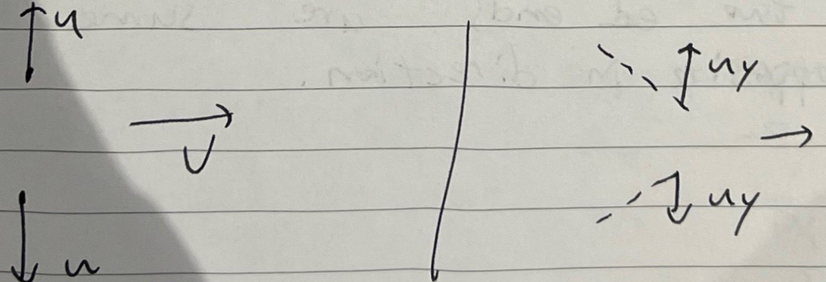
$$= \frac{2u - \frac{2uv^2}{c^2}}{1 - \left(\frac{uv}{c^2}\right)^2} = \cancel{\frac{2u\left(1 - \frac{uv}{c^2}\right)^2 + \frac{4u^2 v}{c^2} - 2u\left(\frac{uv}{c^2}\right)^2 - \frac{2uv^2}{c^2}}{1 - u^2}}$$

$$= \frac{2u\left(1 - \left(\frac{uv}{c^2}\right)^2\right) + 2u\left(\frac{uv}{c^2}\right)^2 - 2u\frac{v^2}{c^2}}{1 - \left(\frac{uv}{c^2}\right)^2}$$

$$= 2u - \frac{2u\frac{v^2}{c^2}\left(1 - \frac{u^2}{c^2}\right)}{1 - \left(\frac{uv}{c^2}\right)^2} \underbrace{\leq}_{\because\ u, v, \leq c} 2u$$

$\therefore$ Horizontal boost from S reduces l

- ~~vertical~~ transverse boost



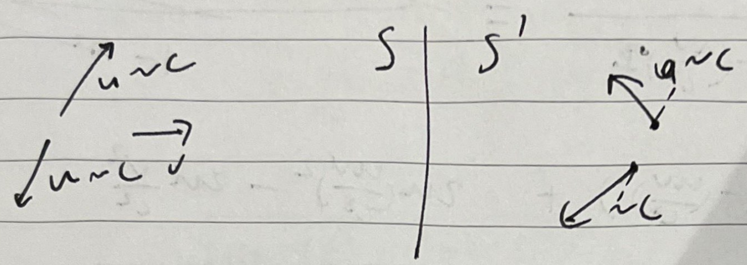
$$u_y' = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \qquad \because u_x = 0 \quad \therefore u_y' = \frac{u_y}{\gamma}$$

If $u_y = u$, then

New $l'' = 2 \times \frac{u}{\gamma} = \frac{2u}{\gamma} \underbrace{\leq}_{\gamma \geq 1} 2u$

$\therefore$ vertical boost also reduces $\dot{l}$

general boost is complicated, but consider extreme case where $u \sim c$



speed of ~~ea~~ each point can-not exceed c

but after boost not all component of velocities are along the spring

$\therefore \quad \dot{l} < 2u \sim 2c$ again.

$\because u \leq c \quad \therefore$ maximum $\dot{l}_{max} = 2c$

the frame is S, where the velocities of two ~~ed~~ ends are same in magnitude but opposite in direction.

$\underline{\text{14B2Q3}}$

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \qquad \underline{B} = \underline{\nabla} \times \underline{A}$$

$$F^{ab} = \partial^a A^b - \partial^b A^a$$

$$\therefore F^{aa} = \partial^a A^a - \partial^a A^a = 0 \qquad (\text{No sum in } a)$$

$$\therefore F^{ab} = -F^{ba} \qquad (\text{antisymmetric})$$

$$F^{0i} = \partial^0 A^i - \partial^i A^0 \qquad A^a = \begin{pmatrix} \phi/c \\ \underline{A} \end{pmatrix}$$

$$= \frac{1}{c}\left(-\frac{\partial}{\partial x_i}\phi - \frac{\partial A^i}{\partial t}\right) \qquad \partial^a = \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} \\ +\frac{\partial}{\partial \underline{x}} \end{pmatrix}$$

$$\cancel{= \frac{1}{c}\left(-\nabla\phi - \frac{\partial A}{\partial t}\right)}$$

$$= +\frac{E^i}{c}$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z$$

$$\cancel{= -\frac{1}{c}\frac{\partial A_y}{\partial t}}$$

$$\therefore F^{ab} = \begin{pmatrix} 0 & +E_x/c & +E_y/c & +E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$\partial_\lambda F^{\lambda b} = -\mu_0 J^b \qquad J^b = \rho_0 U^b = \begin{pmatrix} \rho c \\ \underline{j} \end{pmatrix}$$

$b = 0$: $\partial_\lambda F^{\lambda 0} = -\mu_0 J^0 = -\mu_0 \rho c$

$\because F^{00} = 0 \quad \therefore \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = -\mu_0 \rho c$

$$\partial_\lambda = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \quad (F_{10}, F_{20}, F_{30}) = -\frac{\underline{E}}{c}$$

$$\therefore -\frac{1}{c}\underline{\nabla}\cdot\underline{E} = -\mu_0 \rho c$$

$$\therefore \quad \underline{\nabla}\cdot\underline{E} = \rho\mu_0\frac{1}{\mu_0\varepsilon_0} = \frac{\rho}{\varepsilon_0} \quad (M1).$$

$b = i = 1, 2, 3$

$$\partial_\lambda F^{\lambda i} = -\mu_0 J^i$$

$$-\mu_0 j^1 = \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31}$$

$$= +\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{E_x}{c}\right) + \left(-\frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z}\right)$$

$$= \left(\frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} - \underline{\nabla}\times\underline{B}\right)_x = -\mu_0 j_x$$

$$\therefore \quad \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} - \underline{\nabla}\times\underline{B} = -\mu_0 \underline{j}$$

$$\therefore \underline{\nabla}\times\underline{B} = \mu_0\underline{j} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \quad (M4)$$

$$= \mu_0\underline{j} + \mu_0\varepsilon_0\frac{\partial \underline{E}}{\partial t}$$

$$\because F^{ab} = \partial^a A^b - \partial^b A^a$$

$$\therefore \partial^c F^{ab} + \partial^a F^{bc} + \partial^b F^{ca}$$

$$= \partial^c(\partial^a A^b - \partial^b A^a) + \partial^a(\partial^b A^c - \partial^c A^b)$$
$$+ \partial^b(\partial^c A^a - \partial^a A^c) = 0$$

$$\therefore \quad \partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0$$

$$\rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

$$\rightarrow \quad \underline{\nabla} \cdot \underline{B} = 0 \qquad (M2)$$

$$\partial^0 F^{23} + \partial^2 F^{30} + \partial^3 F^{02} = 0.$$

$$\therefore -\frac{1}{c}\frac{\partial B_x}{\partial t} + \frac{\partial}{\partial y}\left(-\frac{E_z}{c}\right) + \frac{\partial}{\partial z}\left(\frac{E_y}{c}\right) = 0$$

$$\therefore \quad (\underline{\nabla} \times \underline{E})_x = -\frac{\partial B_x}{\partial t}.$$

$$\therefore \underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t} \qquad (M3)$$

$$T^{ab} = \varepsilon_0 c^2 \left(-F^{a\lambda} F^b{}_\lambda - \frac{1}{4} g^{ab} F_{\mu\nu} F^{\mu\nu} \right).$$

$$F^b{}_\lambda = \cancel{g_{\mu\lambda} F} \; g_{\lambda\nu} F^{b\nu}$$

$$= \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \left(\begin{array}{c|ccc} 0 & & \underline{E}/c & \\ \hline & 0 & B_z & -B_y \\ -\underline{E}/c & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{array} \right)$$

$$= \left(\begin{array}{c|ccc} 0 & & -\underline{E}/c & \\ \hline & 0 & B_z & -B_y \\ -\underline{E}/c & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{array} \right).$$

$$\cancel{T^{0b} = \epsilon_0 c^2 \left(-F^{0\lambda} F^{b\lambda} \right)}$$

$$T^{0b} = \epsilon_0 c^2 \left(-F^{0\lambda} F^{b}_{\ \lambda} - \frac{1}{4} g^{0b} F_{\mu\nu} F^{\mu\nu} \right)$$

$$\cancel{F_{\mu\nu} F^{\mu\nu} =}$$

$$F_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & -\vec{E}/c & \\ \hline & 0 & B_z & -B_y \\ \vec{E}/c & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{array} \right)$$

$$\therefore F_{\mu\nu} F^{\mu\nu} = 2B^2 - \frac{2E^2}{c^2}.$$

$$\cancel{T^{0b}} \quad \frac{1}{4} g^{0b} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4} g^{0b} \left(2B^2 - \frac{2E^2}{c^2} \right)$$

$$= \cancel{\frac{1}{4} g^{0b}} \ \frac{1}{2} g^{0b} \left(B^2 - \frac{E^2}{c^2} \right)$$

If $b = 0$, $g^{00} = -1$ $\quad \cancel{\frac{1}{4}} \quad \frac{1}{4} g^{00} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \left(B^2 - \frac{E^2}{c^2} \right)$

If $b \neq 0$ $\quad g^{0b} = 0 \quad \frac{1}{4} g^{0b} F_{\mu\nu} F^{\mu\nu} = 0$

$$-F^{0\lambda} F^{b}_{\ \lambda} = F^{01} F^{b}_{\ 1} + F^{02} F^{b}_{\ 2} + F^{03} F^{b}_{\ 3}$$

$\cancel{\text{If}}$

If $b = 0$, $F^{0\lambda} F^{0}_{\ \lambda} = F^{01} F^{0}_{\ 1} + F^{02} F^{0}_{\ 2} + F^{03} F^{0}_{\ 3}$

$$= -\left(\frac{E_x}{c}\right)^2 - \left(\frac{E_y}{c}\right)^2 - \left(\frac{E_z}{c}\right)^2 = -\frac{E^2}{c^2}$$

$$\therefore \quad T^{00} = \epsilon_0 c^2 \left(\frac{E^2}{c^2} + \frac{1}{2}\left(B^2 - \frac{E^2}{c^2}\right)\right).$$

$$= \underline{\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 c^2 B^2}$$

$\rightarrow$ energy density

$= u$

~~T^{0b}~~

If $b=1$, $\quad F^{0\lambda}F^{1}_{\lambda} = \underbrace{F^{01}F^{1}_{1}}_{0} + F^{02}F^{2}_{1} + F^{03}F^{3}_{1}.$

$$= \left(\frac{E_y}{c}\right)(-B_z) + \left(\frac{E_z}{c}\right)(B_y)$$

$$= \frac{-1}{c}(\underline{E} \times \underline{B})_x$$

$\therefore$ for $b \neq 0$.

~~$T^{01} \to T^{02}$~~ $\quad T^{01} = \epsilon_0 c^2 \left(\frac{+1}{c}(\underline{E} \times \underline{B})_x\right) + 0.$

$$= \epsilon_0 c (\underline{E} \times \underline{B})_x = \frac{1}{c\mu_0}(\underline{E} \times \underline{B})_x$$

$$\therefore (T^{01}, T^{02}, T^{03}) = \underline{\frac{1}{\mu_0}(\underline{E} \times \underline{B})\left(\frac{1}{c}\right)}$$

$\rightarrow$ poynting vector / c

$= \frac{S}{c}$

- $\partial_\lambda T^{\lambda b}$, for $b=0$

$$\partial_\lambda T^{\lambda 0} = \partial_0 T^{00} + \partial_1 T^{10} + \partial_2 T^{20} + \partial_3 T^{30}$$

$$T^{00} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 c^2$$

$F^{a\lambda}F^{b}_{\lambda} =$ ~~$\frac{1}{2}(F^{a\lambda}F^{b}_{\lambda} + F^{b}_{\lambda}F^{a\lambda})$~~

g^{ab} is symmetric

$= F^{a}_{\lambda}F^{b\lambda} = F^{b\lambda}F^{a}_{\lambda}$

$\Rightarrow F^{a\lambda}F^{b}_{\lambda}$ is symmetric

$\therefore T^{ab}$ is symmetric $T^{ab} = T^{ba}$

$$\partial_\lambda = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\therefore \partial_0 T^{00} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\varepsilon_0 c^2 B^2\right) = \frac{1}{c}\frac{\partial u}{\partial t}$$

$$\partial_1 T^{10} = \partial_1 T^{01} = \frac{1}{c}\frac{1}{\mu_0}\frac{\partial}{\partial x}(\underline{E}\times\underline{B})_x = \frac{1}{c}\frac{\partial S_x}{\partial x}$$

$$\therefore \partial_\lambda T^{\lambda 0} = \frac{1}{c}\frac{\partial u}{\partial t} + \frac{1}{c}\underline{\nabla}\cdot\underline{S}$$

~~conservation of energy momentum~~

$$\cancel{\partial_\lambda T^{\lambda 0} = 0 \quad \therefore \frac{\partial u}{\partial t} + \underline{\nabla}\cdot\underline{S} = 0}$$

$$\therefore \varepsilon_0 \underline{E}\cdot\frac{\partial \underline{E}}{\partial t} + \varepsilon_0 c^2 \underline{B}\,\frac{\partial \underline{B}}{\partial t}$$

$$= \frac{1}{\mu_0}\underline{\nabla}\cdot(\underline{E}\times\underline{B}) = \frac{1}{\mu_0}\left[\underline{E}\cdot(\underline{\nabla}\times\underline{B}) + \underline{B}\cdot(\underline{\nabla}\times\underline{E})\right].$$

$$= \frac{1}{c}\left(\frac{\partial u}{\partial t} + \underline{\nabla}\cdot\underline{S}\right)$$

$$\cancel{= \frac{1}{c}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\varepsilon_0\right.}$$

$$= \frac{1}{c}\left(\varepsilon_0 \underline{E}\cdot\frac{\partial \underline{E}}{\partial t} + \varepsilon_0 c^2 \underline{B}\cdot\frac{\partial \underline{B}}{\partial t} + \varepsilon_0 c^2\, \underline{\nabla}\cdot(\underline{E}\times\underline{B})\right)$$

$$= \frac{1}{c}\left[\varepsilon_0 \underline{E}\cdot\frac{\partial \underline{E}}{\partial t} + \varepsilon_0 c^2 \underline{B}\cdot\frac{\partial \underline{B}}{\partial t} + \varepsilon_0 c^2\, \underline{B}\cdot(\underline{\nabla}\times\underline{E}) - \varepsilon_0 c^2\, \underline{E}\cdot(\underline{\nabla}\times\underline{B})\right]$$

$$\underline{\nabla} \times \underline{B} = \frac{\underline{j}}{\varepsilon_0 c^2} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} \quad , \quad \underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\therefore \ \partial_\lambda T^{\lambda 0} = \frac{1}{c}\Big[\varepsilon_0 \underline{E}\cdot\frac{\partial \underline{E}}{\partial t} + \cancel{\varepsilon_0 c^2 \underline{B}\cdot\frac{\partial \underline{B}}{\partial t}}$$

$$\cancel{- \varepsilon_0 c^2 \underline{B}\cdot\frac{\partial \underline{B}}{\partial t}} - \varepsilon_0 c^2 \left(\frac{\underline{j}}{\varepsilon_0 c^2} + \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t}\right)\cdot \underline{E} \Big]$$

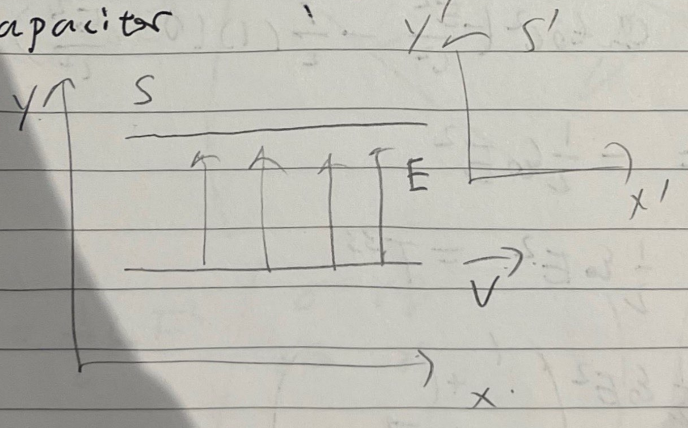
$$= \frac{1}{c}\Big[\cancel{\varepsilon \underline{E}\cdot\frac{\partial \underline{E}}{\partial t}} - \underline{j}\cdot\underline{E} - \cancel{\varepsilon_0 \underline{E}\cdot\frac{\partial \underline{E}}{\partial t}} \Big].$$

$$= -\frac{1}{c}\,\underline{j}\cdot\underline{E}$$

$$\therefore \ \underline{\underline{\partial_\lambda T^{\lambda 0} = -\frac{1}{c}\,\underline{j}\cdot\underline{E}}}$$

→ ~~w~~ work done on the charges

- Capacitor



In S' (rest frame) $\cancel{F^{\alpha'\beta'} = \begin{pmatrix} 0 & [illegible] & [illegible] & [illegible] \\ [illegible] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$

$\cancel{\text{In } S, \ F^{\alpha'\beta'} = \Lambda^{\alpha}{}_{\alpha'} F^{\alpha'\beta'} \Lambda^{\beta}{}_{\beta'}}$

$\cancel{= \Lambda}$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$F' = \cancel{\Lambda^T}\, \Lambda F \Lambda^T$$ Λ (Lorentz transformation)

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore F^{\alpha\beta} = \Lambda^T F' \Lambda = (\Lambda^T)^{\alpha}{}_{\alpha'} F^{\alpha'\beta'} \Lambda_{\beta'}{}^{\beta}$$

$$= \Lambda^{\alpha}{}_{\alpha'} \Lambda^{\beta}{}_{\beta'} F^{\alpha'\beta'}$$

$$\therefore \cancel{T' =}\ T^{\alpha'\beta'} = \cancel{\varepsilon_0 c^2 (} \begin{pmatrix} \frac{1}{2}\varepsilon_0 E^2 & 0 & 0 & 0 \\ 0 & +\frac{1}{2}\varepsilon_0 E^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\varepsilon_0 E^2 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\varepsilon_0 E^2 \end{pmatrix}.$$

because

$$\cancel{F^{\alpha\lambda}}\quad F^{2\lambda} F^{2}{}_{\lambda} = F^{20} F^{2}{}_{0} = +\frac{E^2}{c^2}$$

$$\therefore T^{22} = \varepsilon_0 c^2 \left(-\frac{E^2}{c^2} - \frac{1}{2}(1)\left(0 - \frac{E^2}{c^2}\right)\right)$$

$$= -\frac{1}{2}\varepsilon_0 E^2$$

$$T^{11} = \frac{1}{2}\varepsilon_0 E^2 = T^{33}$$

$$\therefore T^{\alpha'\beta'} = \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} 1 & & & \\ & +1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

$$T^{\alpha'\beta'} = T' = \Lambda T \Lambda^T \qquad \therefore \cancel{T = \Lambda^T T' \Lambda}$$

$$T^{\alpha\beta} = (\Lambda^T)^{\alpha}{}_{\alpha'} T^{\alpha'\beta'} \Lambda_{\beta'}{}^{\beta}$$

$$= \cancel{\Lambda^{\alpha}{}_{\alpha'} T} = \Lambda^{\alpha}{}_{\alpha'} \Lambda^{\beta}{}_{\beta'} T^{\alpha'\beta'}$$

$$\therefore \quad T = \Lambda^{-1} T' (\Lambda^{-1})^{T}$$

$$T^{\alpha\beta} = (\Lambda^{-1})^{\alpha}{}_{\alpha'}\, T^{\alpha'\beta'} (\Lambda^{-1})^{T}{}_{\beta'}{}^{\beta}$$

$$= (\Lambda^{-1})^{\alpha}{}_{\alpha'} (\Lambda^{-1})^{\beta}{}_{\beta'}\, T^{\alpha'\beta'} \qquad \Lambda^{-1} = \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\therefore \quad T^{\alpha\beta} = \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \cancel{\gamma\beta} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ \gamma\beta & -\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} \gamma^2+\gamma^2\beta^2 & -\gamma^2\beta-\gamma^2\beta & 0 & 0 \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

$$T = \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \cancel{\frac{1}{2}\varepsilon_0 E^2} \Bigg($$

$$= \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} \gamma^2+\gamma^2\beta^2 & 2\gamma^2\beta & & \\ 2\gamma^2\beta & \gamma^2+\gamma^2\beta^2 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

$$\gamma^2(1+\beta^2) = \frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} = \frac{c^2+v^2}{c^2-v^2}$$

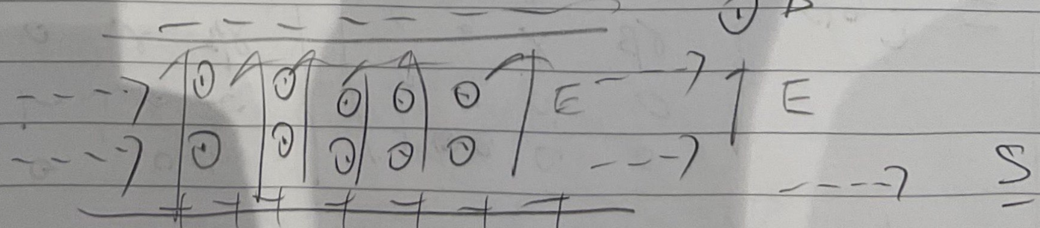
$$2\gamma^2\beta = 2\frac{v/c}{(\sqrt{1-v^2/c^2})^2} = \frac{2v/c}{1-\frac{v^2}{c^2}}$$

$$= \frac{2vc}{c^2-v^2}$$

$$= \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} \frac{c^2+v^2}{c^2-v^2} & \frac{2vc}{c^2-v^2} & & \\ \frac{2vc}{c^2-v^2} & \frac{c^2+v^2}{c^2-v^2} & & \\ & & -1 & \\ & & & 1 \end{pmatrix}.$$

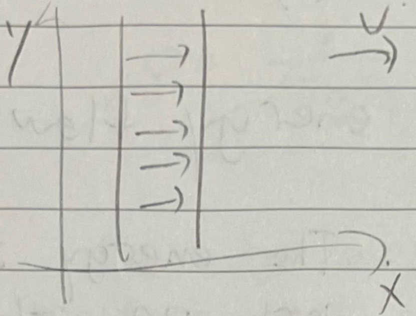
Poynting vector $\frac{\underline{S}}{c} = \left(\frac{2vc}{c^2-v^2}, 0, 0\right). \times \frac{1}{2}\varepsilon_0 E^2$

$$\therefore \underline{S} = \left(\frac{\cancel{2}vc^2}{c^2-v^2}\varepsilon_0 E^2, 0, 0\right) = \cancel{\varepsilon_0}\, \varepsilon_0 E^2\gamma^2 v\,\hat{\underline{x}}$$



~~energy flow from left side into capacitor and a flow out from right side.~~

- energy is transported by the field itself.
the transported energy is along the motion of capacitor.
the energy flows by the ~~flet~~ flow of field.
Similarly, for capacitor in yz plane



$$T' = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \frac{1}{2}\varepsilon_0 E^2$$

$$T = \Lambda^{-1} T' \Lambda^{-1} =$$

$$\frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & & \\ -\gamma\beta & -\gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & & \\ 0 & \gamma^2(\beta-1) & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

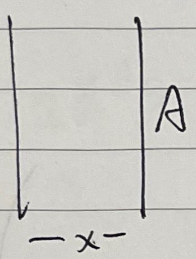
$$\gamma^2(1-\beta^2) = \frac{1}{\cancel{1-\frac{v^2}{c^2}}}\left(\cancel{1-\frac{v^2}{c^2}}\right) = 1.$$

$\therefore \rho$

$$\therefore T = \frac{1}{2}\varepsilon_0 E^2 \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$ unchanged.

$\underline{S} = 0$ No energy flow. ~~between~~

in the field. The energy is transported by plates doing work on the field.



$$C = \frac{\varepsilon_0 A}{x} \qquad U = \frac{1}{2}\left(\frac{\varepsilon_0 A}{x}\right)^{-1} Q^2$$

$$= \frac{Q^2}{2\varepsilon_0 A} x$$

Force $f = \frac{dW}{dx} = \frac{Q^2}{2\varepsilon_0 A}$

E field $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0}$

$$\therefore f = \frac{1}{2} Q E = \frac{1}{2}\varepsilon_0 A E^2$$

$\underline{f} \cdot \underline{V} =$ ~~the~~ rate of doing work

$$= \frac{1}{2} Q E V = \frac{1}{2}\varepsilon_0 E^2 A V$$

$$= \frac{d}{dt}\Big[\underbrace{\tfrac{1}{2}\varepsilon_0 E^2}_{\text{energy of field per volume}} \underbrace{V}_{\text{volume}}\Big]$$

– this is the rate of transporting energy density $\frac{1}{2}\varepsilon_0 E^2$ along $\underline{\hat{x}}$

14B2Q4

charged particle charge q at rest

$$\phi = \frac{q}{4\pi\varepsilon_0 r_{sf}} \qquad \underline{A} = 0$$

(r_{sf} = distance from source event to field event)

General solutions to maxwells equations

$$\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\underline{r_{sf}}, t - \frac{r_{sf}}{c})}{r_{sf}} d^3 \underline{r_{sf}}$$

$$\underline{A} = \frac{1}{4\pi\varepsilon_0 c}\int \frac{\underline{j}(\underline{r_{sf}}, t - \frac{r_{sf}}{c})}{r_{sf}} d^3 \underline{r_{sf}}$$

$\phi, \underline{A}$ only depend on the displacement from source event to field event, and the velocity of charge at source event (contained in $\underline{j}$)

$\therefore$ The covariant form of A (4 vector) must only depend on the 4-velocity of charge at source event and the 4-displacement from source event to field event.

At frame S' (rest frame of charge)

$$A = \begin{pmatrix} \phi/c \\ \underline{A} \end{pmatrix} = \frac{q}{4\pi\varepsilon_0 c r_{sf}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{q}{4\pi\varepsilon_0 c^2 r_{sf}} \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} = U \quad \text{at rest frame } (S') \quad \therefore \frac{U}{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ at } S'$$

Consider $-R \cdot U$. $\quad R = 4$ displacement

$$= (ct_{sf}, \underline{r_{sf}})$$

$$t_{sf} = \frac{r_{sf}}{c} \quad \therefore R = (r_{sf}, \underline{r_{sf}})$$

field ~~travels~~ effect travels at c

$$-R \cdot U = -\begin{pmatrix} r_{sf} \\ \underline{r_{sf}} \end{pmatrix} \cdot \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$$

$$= \gamma r_{sf} c - \gamma \underline{r_{sf}} \cdot \underline{v}$$

At rest frame $\underline{v} = 0$, $\gamma = 1$ $\therefore -R \cdot U = r_{sf} c$

$\therefore$ We have in S'

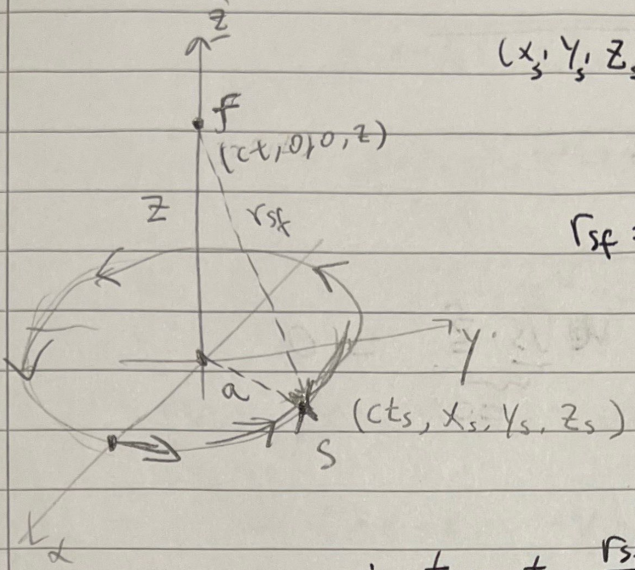
$$A = \frac{q}{4\pi\varepsilon_0} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \left(\frac{1}{r_{sf} c} \right)$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{U/c}{[-R \cdot U]}$$

$\because A$ only depend on U, R in any frame

$\therefore$ This is a genuinely covariant form of A in any frame.

- $\underline{E}$ of circular motion charge :



$(x_s, y_s, z_s) = (a\cos\omega t_s, a\sin\omega t_s, 0)$

$r_{sf} = \sqrt{a^2+z^2}$

$\therefore t_{sf} = \frac{r_{sf}}{c}$ $\quad t - t_s = t_{sf}$

$$\therefore t_s = t - \frac{r_{sf}}{c} = t - \frac{\sqrt{a^2+z^2}}{c}$$

R At source time

$$R = \begin{pmatrix} r_{sf} \\ \underline{r_{sf}} \end{pmatrix} = \begin{pmatrix} \sqrt{a^2+z^2} \\ -x_s \\ -y_s \\ z \end{pmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\frac{v_s^2}{c^2}}}$$

$$U = \begin{pmatrix} \gamma c \\ \gamma \underline{v_s} \end{pmatrix} \qquad \therefore -R \cdot U = \gamma c\sqrt{a^2+z^2} - \gamma \underline{v_s} \cdot (-x_s, -y_s, z)$$

$$\therefore A = \frac{q}{4\pi\varepsilon_0 c} \frac{(1) \quad (\gamma)}{[\gamma c\sqrt{a^2+z^2} - \gamma \underline{v_s} \cdot (-x_s, -y_s, z)]} \begin{pmatrix} c \\ \underline{v_s} \end{pmatrix}$$

~~$\because \underline{v_s} \perp$~~ $\underline{v_s} = (-\omega a\sin\omega t_s, \omega a\cos\omega t_s, 0)$
$= \omega(-y_s, x_s, 0)$

$\therefore \underline{v_s} \cdot (-x_s, -y_s, z) = 0$

~~$\therefore A = \frac{q}{4\pi\varepsilon_0 c^2} \frac{1}{\sqrt{a^2+z^2}} (c$~~

$$\therefore\ A = \frac{q}{4\pi\varepsilon_0 c^2} \frac{1}{\sqrt{a^2+z^2}} \begin{pmatrix} c \\ \underline{V_s} \end{pmatrix}$$

$$E_z = -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t}$$

$$A_z = \frac{q}{4\pi\varepsilon_0 c^2} \frac{1}{\sqrt{a^2+z^2}} \underbrace{\underline{V_s} \cdot \hat{z}}_{=0} = 0$$

$$\therefore\ E_z = -\frac{\partial \phi}{\partial z}$$

$$\phi = c \times \frac{q}{4\pi\varepsilon_0 \cancel{c^2}} \frac{1}{\sqrt{a^2+z^2}} c$$

$$= \frac{q}{4\pi\varepsilon_0 \sqrt{a^2+z^2}} = \frac{q}{4\pi\varepsilon_0}(a^2+z^2)^{-\frac{1}{2}}$$

$$\therefore\ E_z = -\frac{\partial \phi}{\partial z} = \frac{\cancel{-}q}{4\pi\varepsilon_0}\left(\cancel{-}\frac{1}{\cancel{2}}\right)(a^2+z^2)^{-\frac{3}{2}}(\cancel{2}z)$$

$$= \frac{qz}{4\pi\varepsilon_0 (a^2+z^2)^{3/2}}$$ ✓

- E_x ??

$r_F = \cancel{\sqrt{(\Delta x - x_s)^2}}$

Field event at $(\Delta x, 0, z)$

$\cancel{A = \frac{q}{4\pi\varepsilon_0 c}}$ ~~r~~

$$r_{sf} = \left|(\Delta x, 0, z) - (x_s, y_s, 0)\right|$$

$$= \sqrt{(\Delta x - x_s)^2 + y_s^2 + z^2}$$

$$= \sqrt{\underbrace{x_s^2 + y_s^2}_{a^2} + z^2 - 2x_s\Delta x + \underbrace{\Delta x^2}_{\sim 0}}$$

$$= \sqrt{a^2 + z^2 - 2x_s\Delta x} \simeq \sqrt{a^2+z^2}\left(1 - \frac{2x_s}{a^2+z^2}\Delta x\right)^{+\frac{1}{2}}$$

$$= \sqrt{a^2+z^2}\left(1 - \frac{x_s\Delta x}{a^2+z^2}\right)$$

$$\underline{r_{sf}} = (\Delta x, 0, z) - (x_s, y_s, 0)$$

$$= (\Delta x - x_s, -y_s, z)$$

$$\therefore \underline{v_s}\cdot\underline{r_{sf}} = \omega(-y_s, x_s, 0)\cdot(\Delta x - x_s, -y_s, z)$$

$$= \omega[-y_s\Delta x + \cancel{y_s x_s} - \cancel{x_s y_s} + 0]$$

$$= -\omega y_s \Delta x$$

$$\therefore A = \frac{q}{4\pi\varepsilon_0}\frac{U/c}{(-R\cdot U)} = \frac{q}{4\pi\varepsilon_0 c}\frac{(1)\,\gamma}{\gamma(c r_{sf} - \gamma \underline{v_s}\cdot\underline{r_{sf}})}\begin{pmatrix} c \\ \underline{v_s} \end{pmatrix}$$

$$= \frac{q}{4\pi\varepsilon_0 c}\frac{1}{c\sqrt{a^2+z^2} - \frac{x_s c\Delta x}{\sqrt{a^2+z^2}} + \omega y_s\Delta x}\begin{pmatrix} c \\ \underline{v_s} \end{pmatrix}$$

$$= \frac{q}{4\pi\varepsilon_0 c^2\sqrt{a^2+z^2}}\left(1 + \frac{x_s\Delta x}{a^2+z^2} - \frac{\omega y_s\Delta x}{c\sqrt{a^2+z^2}}\right)\begin{pmatrix} c \\ \underline{v_s} \end{pmatrix}$$

$$E_x = -\frac{\partial\phi}{\partial x} - \frac{\partial A_x}{\partial t} = -\frac{\partial\phi}{\partial(\Delta x)} - \frac{\partial A_x}{\partial t}$$

$$V_s\cdot\hat{\underline{x}} = -\omega y_s = -\omega a\sin\omega t_s$$

$$-\frac{\partial(V_s\cdot\hat{\underline{x}})}{\partial t} = \omega^2 a\cos(\omega t_s)$$

treat $\Delta x = 0$ when doing time derivative!

$$= \frac{q}{4\pi\varepsilon_0 c^2 \sqrt{a^2+z^2}} \left[\underbrace{\frac{-x_s c^2}{a^2+z^2} + \frac{\omega y_s c^2}{c\sqrt{a^2+z^2}}}_{-\frac{\partial \phi}{\partial (\Delta x)}} + \underbrace{\omega^2 a \cos \omega t_s}_{-\frac{\partial A_x}{\partial t}} \right]$$

$(x_s = a\cos\omega t_s, \quad y_s = a\sin\omega t_s$

$$= \frac{qa}{4\pi\varepsilon_0 c^2 \sqrt{a^2+z^2}} \left[\left(\omega^2 - \frac{c^2}{a^2+z^2} \right) \cos\omega t_s + \frac{\omega c}{\sqrt{a^2+z^2}} \sin\omega t_s \right]$$