2761

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B1: II. SYMMETRY AND RELATIVITY

TRINITY TERM 2013

Wednesday, 12 June, 2.30 pm – 4.30 pm

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Throughout this paper, units in which $c = 1$ are adopted.

The Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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1. Write down a formula for the rest mass of a system composed of two particles whose 4-momenta are P_1, P_2. [2]

(a) A neutral particle is produced in a high-energy interaction. It then travels 0.9 mm in a direction at approximately $+10°$ to the axis of a detector, before decaying into K^+, π^- particles. In the laboratory, the K^+ and π^- emerge at angles $+8.343°$ and $+33.99°$ to the detector axis, with momenta 9541 MeV/c and 1625 MeV/c respectively. Find a more accurate value for the direction of travel of the neutral particle, the rest mass of the neutral particle, and its proper lifetime before this particular decay. [10]

Find the momentum of either decay product (K^+ or π^-) in the CM frame, and the direction in which it was emitted in CM frame. [6]

(b) Show generally that, when a particle of rest mass M decays into a pair of particles of rest masses m_1, m_2, the 3-momentum p of either decay product in the CM frame satisfies

$$p^2 = \frac{M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}{4M^2}.$$

Use this to confirm your calculation in part (a). [7]

θ_{K^+} , θ_{π^-} , θ_M

2. A particle undergoing hyperbolic motion has a worldline given by

$$x^2 - t^2 = L^2$$

where L is a constant. Find the speed and acceleration of such a particle as functions of x. Find the relationship between the Lorentz factor and x, and hence show that the proper acceleration is constant. [You may quote the relationship between acceleration and proper acceleration for rectilinear motion.] [7]

Explain what is meant by the terms *field event*, *source event* and *projected position* in a calculation of the electromagnetic field of an arbitrarily moving particle. Illustrate your answer by drawing on a spacetime diagram the worldline given above, and indicating on the diagram the field event, source event and projected position, for the case where the field event $(t_f, \mathbf{x}_f)$, is on the x axis at $x_f > L$ and $t_f = 0$. [5]

Show that, for such a field event, the source event is at

$$t_s = \frac{L^2 - x_f^2}{2x_f}\,, \qquad x_s = \frac{L^2 + x_f^2}{2x_f}\,.$$

$\frac{x_s}{L}$ $\frac{L}{x_s}$ $= L$ $\frac{L^2}{x_s} = x_p$

Hence find the projected position, and the electric field at the field event. [You may assume that the electric field of a uniformly moving charge is given by $E = q/(4\pi\epsilon_0 r^2\gamma^2)$ for points on the line of motion of the charge.] [9]

Obtain the ratio between the electric field at $(t_f, x_f) = (0, 2L)$ sourced by this charge, and the field which would be produced at that position by a similar charge permanently at rest at $x = L$. Comment briefly on the electric field at $t_f = 0$ for field events at small positive x_f (that is, $0 < x_f \ll L$). [4]

3. A two-component spinor s may be used to represent the 4-momentum P of a massless particle by using the relation $\mathsf{P}^a = s^\dagger \sigma^a s$, where σ^a are the Pauli spin matrices.

(a) Obtain the 4-momentum for such a particle in each of the following cases:

$$s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad s = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad s = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \quad s = \begin{pmatrix} e^{\mathrm{i}|\alpha|} \\ 0 \end{pmatrix}.$$ [4]

Under a change of reference frame, the spinor transforms as $s' = \Lambda s$ where

$$\Lambda = e^{\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2 - \boldsymbol{\sigma}\cdot\boldsymbol{\rho}/2}.$$

in which θ is the angle through which the coordinate system is rotated, and ρ is the rapidity.

(b) Show that $\exp(\mathrm{i}\sigma^a\theta/2) = I\cos(\theta/2) + \mathrm{i}\sigma^a \sin(\theta/2)$. By starting from a spinor directed along the z axis, and applying suitable rotations, confirm your answer to the first three examples in part (a). [8]

(c) By applying a rotation to a previous result, or otherwise, find a spinor representing a particle whose energy is E and whose momentum is along the direction $(1, 1, 0)$ in some inertial frame F. For convenience, set the overall phase (the flag angle) such that the first component of the spinor is a real number. Find the spinor components for this same particle when it is observed in a frame moving at velocity $(15/17)c$ in the z direction relative to F. Hence obtain the transformed 4-momentum, and use it to obtain the angle between the particle velocity and the z axis in the new frame. [13]

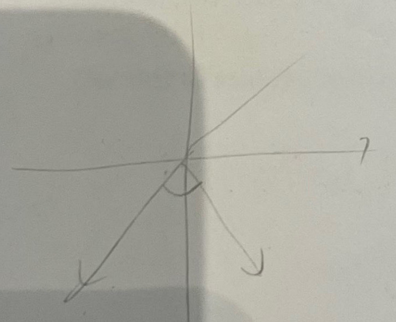
-22.5°

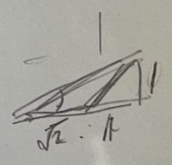
$e^{-i(\alpha+\phi)}$

$\alpha+\phi=0$

$\alpha-\phi=-\pi$

$\alpha=$





$1-\Omega^2R^2 = (1-\Omega x)^2$

$1-\Omega x = \sqrt{1-\Omega^2R^2}$ $\qquad x = \frac{1}{\Omega}\left(1-\sqrt{1-\Omega^2R^2}\right)$

$\therefore\ \Omega x = 1-\sqrt{1-\Omega^2R^2}$ $\qquad \simeq \frac{1}{\Omega}\left(1-\left(1-\frac{1}{2}\Omega^2R^2\right)\right)$

$= \frac{1}{\Omega}\,\frac{1}{2}\Omega(R^2-z^2)$

4. Frames S and S′ are in standard configuration with relative speed β. A source at rest in S′ emits a plane wave of angular frequency ω_0 a direction making angle θ_0 to the x' axis and lying in the $x'y'$ plane. Find the frequency ω observed in frame S as a function of ω_0 and θ, where θ is the angle between the observed ray and the x axis in frame S, and show that

$$\cos\theta_0 = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}\,.$$

[6]

Given that the intensity of a plane wave transforms as $I/I_0 = (\omega/\omega_0)^2$, show that when the emission is isotropic in the rest frame S′, the brightness (power per unit solid angle $\tilde{\Omega}$) of the light observed in S is related to that in S′ by

$\frac{dP_0}{d\Omega_0} = \left(\frac{\omega_0}{\omega}\right)^4 \times \cos\theta$

$$\frac{\mathrm{d}P}{\mathrm{d}\tilde{\Omega}} = \frac{(1-\beta^2)^2}{(1-\beta\cos\theta)^4}\,\frac{\mathrm{d}P_0}{\mathrm{d}\tilde{\Omega}_0}\,.$$

$1-\Omega^2R^2$ $\qquad \frac{\sqrt{1-\Omega^2R^2}}{1-\Omega x}$

[7]

A rapidly rotating star is modelled approximately as a sphere of radius R rotating rigidly at angular frequency Ω about the z axis of an inertial frame S. An observer is far away in the y direction, at rest relative to the centre of the star. Consider the material on the surface of the star at coordinates $(x, y, z) = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$ in frame S. Show that if this material emits radiation of angular frequency ω_0 in its own rest frame, then the frequency observed by the distant observer is (for $c = 1$)

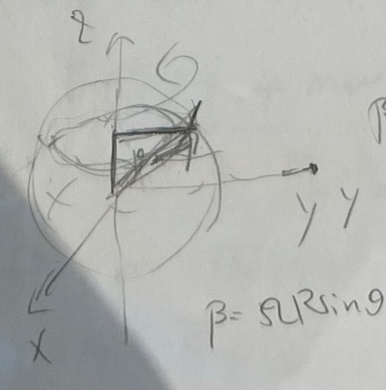
$$\omega = \omega_0\frac{\sqrt{1-\Omega^2(R^2-z^2)}}{1-\Omega x}\,.$$

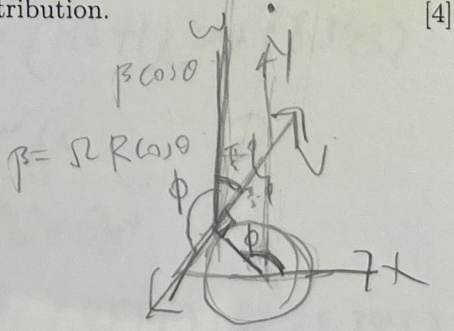
$\frac{\sqrt{1-\beta^2}}{(1-\beta\cos\theta)}$

Draw a circle representing the glowing disk seen by the observer. Roughly indicate on your diagram the locus of points for which $\omega = \omega_0$, and indicate which region has $\omega > \omega_0$, which has $\omega < \omega_0$. If the brightness of the observed disk is uniform, roughly estimate the average observed Doppler shift $\Delta\lambda$ for a star with $R = 2\times10^6$ km, $\Omega = 3\times10^{-4}$ rad s^{-1} whose surface emits at wavelength $\lambda = 500$ nm (no integration is required; a rough estimate of the magnitude and sign suffices). [8]

In fact the disk of light falling on a plane detector is not expected to be of uniform brightness (although for a non-rotating star it would be). Explain why not. A detector on Earth typically cannot resolve the disk, but this brightness distribution will affect the observed frequency distribution. Discuss qualitatively how this modifies your estimate for the shift in the average of the observed frequency distribution. [4]

$\frac{\omega}{\omega_0} \simeq \frac{\sqrt{1-\Omega^2R^2}}{\sqrt{1-\Omega^2R^2}}$



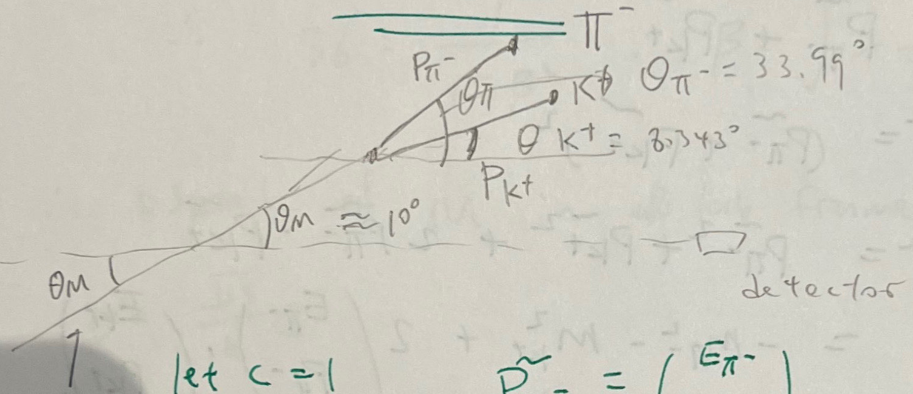


B2 2013 ①

$\eta = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ $\quad P_1 = \begin{pmatrix} E_1/c \\ \underline{p_1} \end{pmatrix}$ $\quad P_2 = \begin{pmatrix} E_2/c \\ \underline{p_2} \end{pmatrix}$

rest mass $\quad -M^2c^2 = (P_1 + P_2)\cdot(P_1+P_2).$

$\therefore \quad M = \sqrt{-\dfrac{(P_1+P_2)^2}{c^2}}$



let $c = 1$ $\quad \tilde{P}_{\pi^-} = \begin{pmatrix} E_{\pi^-} \\ \underline{p_{\pi^-}} \end{pmatrix}$ $\quad \tilde{P}_{K^+} = \begin{pmatrix} E_{K^+} \\ \underline{p_{K^+}} \end{pmatrix}.$

$\tilde{P}_M = \begin{pmatrix} E_M \\ \underline{p_M} \end{pmatrix}$ $\quad$ ~~$\theta_{\pi^-} = 33$~~ $\quad \theta_{K^+} = 8.343°$

$\theta_{\pi^-} = 33.99°$

Conservation of momentum

$p_M \cos\theta_M = p_{\pi^-}\cos\theta_{\pi^-} + p_{K^+}\cos\theta_{K^+}$

$= (1625)\cos(33.99) + (9541)\cos(8.343)$

$= 10787.4 \text{ MeV}/c$ ①

$p_M \sin\theta_M = p_{\pi^-}\sin\theta_{\pi^-} + p_{K^+}\sin\theta_{K^+}$

$= (1625)\sin(33.99) + (9541)\sin(8.343)$

$= 2292.84 \text{ MeV}/c.$ ②

$\therefore \quad \frac{②}{①} \Rightarrow \tan\theta_m = 0.21255$

$\rightarrow \quad \theta_m = 12^\circ$

$\therefore \quad P_m = 11028 \text{ MeV}$

- rest mass

$$\tilde{P}_m = \tilde{P}_{\pi^-} + \tilde{P}_{K^+}$$

$$\therefore \quad \tilde{P}_m^{\,2} = (\tilde{P}_{\pi^-} + \tilde{P}_{K^+})^2$$

$$\therefore \quad -M^2 = \tilde{P}_{\pi^-}^{\,2} + \tilde{P}_{K^+}^{\,2} + 2\tilde{P}_{\pi^-}\cdot\tilde{P}_{K^+}$$

$$= -m_{\pi}^2 - m_{K^+}^2 + 2\begin{pmatrix}E_{\pi^-}\\ P_{\pi^-}\end{pmatrix}\cdot\begin{pmatrix}E_{K^+}\\ P_{K^+}\end{pmatrix}$$

$$= -m_{\pi}^2 - m_{K^+}^2 + 2\left(-E_{\pi^-}E_{K^+} + P_{\pi^-}P_{K^+}\cos(\theta_{K^+} - \theta_{\pi^-})\right)$$

$\cancel{= m_\pi}$ $\because E^2 = p^2 + m^2$

$$\therefore \quad M = \left(m_{\pi^-}^2 + m_{K^+}^2 + 2\sqrt{m_\pi^2 + P_\pi^2}\sqrt{m_{K^+}^2 + P_{K^+}^2} - 2P_{K^+}P_{\pi^-}\cos(\theta_{K^+} - \theta_{\pi^-})\right)^{\frac{1}{2}}$$

(in MeV)

$$= \left((139.57)^2 + (493.7)^2 + 2\left((139.57)^2 + (1625)^2\right)^{\frac{1}{2}}\left((493.7)^2 + (9541)^2\right)^{\frac{1}{2}} - 2(1625)(9541)\cos(33.99 - 8.343)\right)^{\frac{1}{2}}$$

$$= 1864 \text{ MeV}/c^2$$

For M: $M = 1864\ \text{MeV}/c^2$

$P_M = 11028\ \text{MeV}$

$\therefore E_M = (P_M^2 + M^2)^{\frac{1}{2}} = \underline{11184\ \text{MeV}}$

$\therefore \gamma = \frac{E_M}{M} = \frac{11184}{1864} = 6 = \frac{1}{\sqrt{1-\beta^2}}$ $(\beta = \frac{v}{c})$

$\therefore 36 = \frac{1}{1-\beta^2}$ $\therefore \beta = \sqrt{1-\frac{1}{36}} = \frac{\sqrt{35}}{6}$

$\therefore$ speed of M in lab frame S is

$v = \frac{\sqrt{35}}{6}c$

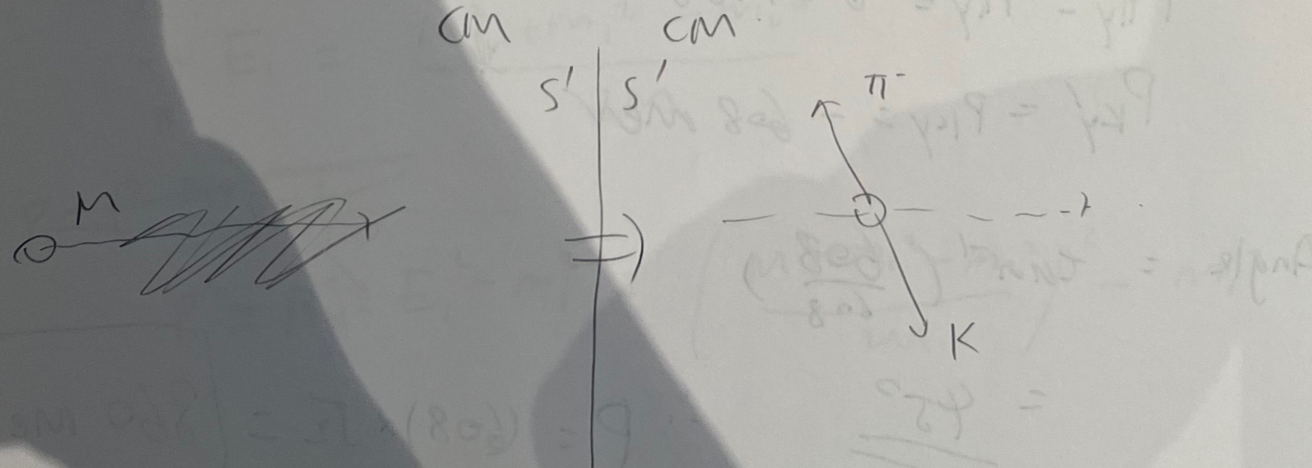
travel time in S frame:

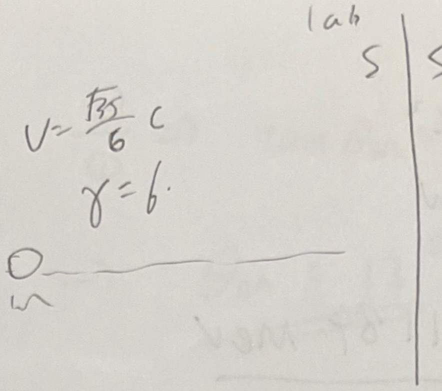
$\Delta t = \frac{\Delta x}{v} = \frac{0.9\times10^{-3}\,m}{3\times10^{8}\,m/s}\left(\frac{6}{\sqrt{35}}\right) = 3.04\times10^{-12}\,s$

in rest frame of M (frame S'):

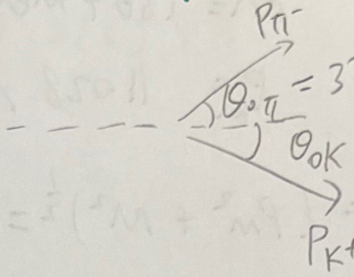
by time dilation, proper lifetime

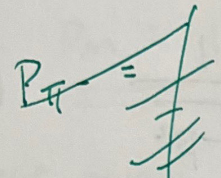
$\Delta T = \frac{\Delta t}{\gamma} = \frac{\Delta t}{6} = \underline{\underline{5.1\times10^{-13}\,s}}$





$E_{\pi} = (m_\pi^2 + P_\pi^2)^{\frac{1}{2}} = 1630.98$
$E_K = (m_K^2 + P_K^2)^{\frac{1}{2}} = 9553.76$





$P_{\pi x} = P_{\pi^-} \cos(21.99) = \cancel{8846.88}\ 1506.78\ \frac{MeV}{c}.$

$P_{Kx} = P_{K^+} \cos(3.657) = 9521.57\ \frac{MeV}{c}$

$P_{\pi y} = -P_{Ky} = P_{\pi^-} \sin(21.99) = 608.47\ MeV/c \simeq 608\ MeV/c$

~~tra~~ Lorentz transformation from S' to S

$$\Rightarrow \quad P_{\pi x}' = \gamma\left(P_{\pi x} - \frac{v E_{\pi}}{c^2}\right).$$

$$= 6\left(1506.78 - \frac{\sqrt{35}}{6}\, 1630.98\right).$$

$$= -608.328\ \frac{MeV}{c}.$$

$$P_{Kx}' = \gamma\left(P_{Kx} - \frac{vE_K}{c^2}\right) = 6\left(9521.57 - \frac{\sqrt{35}}{6}\, 9553.76\right)$$

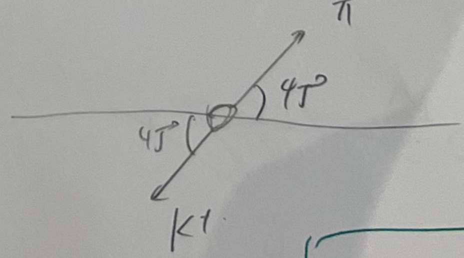
$$= 608.6\ \frac{MeV}{c}$$

(they should be the same in magnitude).

$$\therefore \quad -P_{\pi x}' = P_{Kx}' = \boxed{608\ \frac{MeV}{c}}$$

$P_{\pi y}' = P_{\pi y} = 608\ MeV$

$P_{Ky}' = P_{Ky} = -608\ MeV$

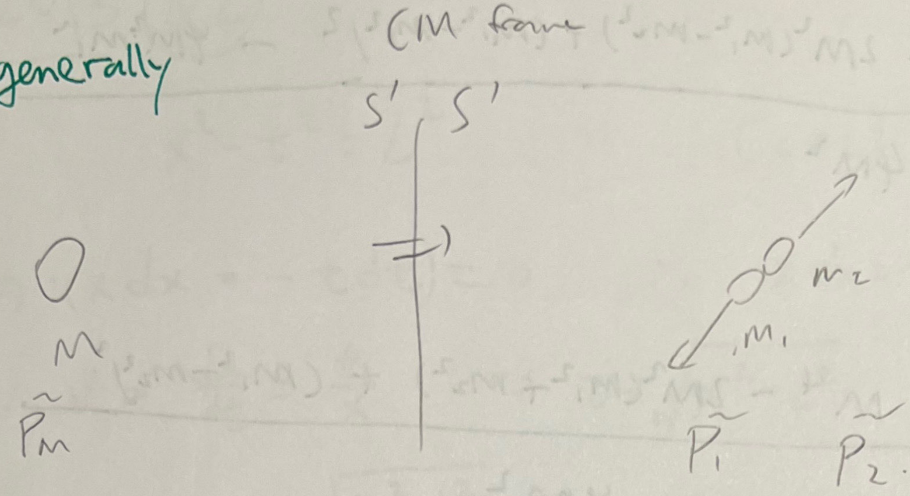


$$\text{Angle} = \tan^{-1}\left(\frac{608}{608}\right)$$

$$= \underline{45°}$$

$$\therefore\ p = (608) \times \sqrt{2} = \boxed{860\ MeV/c}$$

- generally



$$\cancel{\tilde{P}_M^{\,2} = (\tilde{P}_1 + \tilde{P}_2)^2}$$

$$\cancel{\therefore -M^2 = -m_1^2 - m_2^2 + 2\tilde{P}_1 \cdot \tilde{P}_2}$$

$$\cancel{-M^2 = -m_1^2 - m_2^2 -}$$

$$\tilde{P}_M = \tilde{P}_1 + \tilde{P}_2 \qquad \therefore \tilde{P}_2^{\,2} = (\tilde{P}_M - \tilde{P}_1)^2$$

$$\therefore \cancel{P} - m_2^2 = -M^2 - m_1^2 - 2\tilde{P}_M \cdot \tilde{P}_1$$

$$\tilde{P}_M \cdot \tilde{P}_1 = \begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ \underline{p}_1 \end{pmatrix} = -ME_1.$$

$$\therefore -m_2^2 = -M^2 - m_1^2 + 2ME_1$$

$$\therefore E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}$$

$$\cancel{p = p_1 = \sqrt{E_1^2}}$$

$$p^2 = p_1^2 = E_1^2 - m_1^2 = \left(\frac{M^2 + m_1^2 - m_2^2}{2M}\right)^2 - m_1^2.$$

B2

$$= \frac{M^4 + 2M^2(m_1^2 - m_2^2) + (m_1^2 - m_2^2)^2 - 4M^2 m_1^2}{4M^2}$$

$$= \frac{M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}{4M^2}$$

in part (a)

$$P^* = \left[\frac{1864^4 - 2 \cdot 1864^2 (139.57^2 + 493.7^2) + (493.7^2 - 139.57^2)^2}{4 \times 1864^2} \right]^{\frac{1}{2}}$$

$$= \cancel{868\ \text{meV}} \quad \boxed{860\ \text{meV}/c}$$

B2 2013 (Q2)

$$x^2 - t^2 = L^2 \qquad (c=1)$$

$$\Rightarrow 2(x\,dx - t\,dt) = 0 \quad \therefore \quad x\,dx = t\,dt.$$

$$\therefore v = \frac{dx}{dt} = \frac{t}{x} \quad \because \quad t = \sqrt{x^2 - L^2}$$

$$\therefore \quad v = \frac{\sqrt{x^2-L^2}}{x} \qquad \text{(speed } v\text{)}.$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{t}{x}\right) = \frac{x - t\frac{dx}{dt}}{x^2} = \frac{1}{x} - \frac{t}{x^2}\frac{dx}{dt}.$$

$$= \frac{1}{x} - \cancel{}\frac{v^2}{x} = \frac{1-v^2}{x}.$$

$$= \frac{1}{x}\left(1 - \frac{x^2-L^2}{x^2}\right) = \frac{1}{x}\left(+\frac{L^2}{x^2}\right) = \frac{L^2}{x^3}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1 - \frac{x^2-L^2}{x^2}}} = \frac{1}{\sqrt{\frac{L^2}{x^2}}} = \frac{x}{L}$$

in rest frame ~~A~~ 4 acceleration $A' = \begin{pmatrix} 0 \\ a_0 \end{pmatrix}$.

$$\therefore \quad A \cdot A = A' \cdot A' = a_0^2$$

$$A = \cancel{\frac{d}{d\tau}}\frac{dU}{d\tau} = \frac{d}{d\tau}\begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix} = \cancel{\frac{d}{dt}\frac{dt}{d\tau}\gamma} \frac{dt}{d\tau}\frac{d}{dt}\begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}.$$

$$= \gamma \frac{d}{dt}\begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$$

$$\frac{d}{dt}(\gamma c) = c\frac{d\gamma}{dt} = c\frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{\underline{v}\cdot\underline{v}}{c^2}}}\right) = \frac{\gamma^3}{c^2}\underline{v}\cdot\dot{\underline{v}}\cdot c$$

$$= \frac{\gamma^3}{c}\underline{v}\cdot\dot{\underline{v}}$$

$$\frac{d}{dt}(\gamma \underline{v}) = \gamma \frac{d\underline{v}}{dt} + \underline{v}\frac{d\gamma}{dt}$$

$$= \gamma \dot{\underline{v}} + \frac{\gamma^3}{c^2}(\underline{v}\cdot\dot{\underline{v}})\underline{v}$$

$$\therefore \quad A = \begin{pmatrix} \frac{\gamma^4}{c}\,\underline{v}\cdot\dot{\underline{v}} \\ \gamma^2\dot{\underline{v}} + \frac{\gamma^4}{c^2}(\underline{v}\cdot\dot{\underline{v}})\underline{v} \end{pmatrix}$$

$$\therefore \quad A\cdot A = -\left(\frac{\gamma^4}{c^2}\underline{v}\cdot\dot{\underline{v}}\right)^2 + \left(\gamma^2\dot{\underline{v}} + \frac{\gamma^4}{c^2}(\underline{v}\cdot\dot{\underline{v}})\underline{v}\right)^2$$

$$= -\left(\frac{\gamma^4}{c^2}\underline{v}\cdot\dot{\underline{v}}\right)^2 + \gamma^4\dot{v}^2 + 2\gamma^6\frac{(\underline{v}\cdot\dot{\underline{v}})^2}{c^2} + \left(\frac{\gamma^4}{c^2}(\underline{v}\cdot\dot{\underline{v}})\underline{v}\right)^2$$

$$= \gamma^4\dot{v}^2 + \frac{2\gamma^6(\underline{v}\cdot\dot{\underline{v}})^2}{c^2} - \frac{\gamma^8}{c^4}(\underline{v}\cdot\dot{\underline{v}})^2\underbrace{\left(1-\frac{v^2}{c^2}\right)}_{\frac{1}{\gamma^2}}$$

$$= \gamma^4\dot{v}^2 + \frac{\gamma^6(\underline{v}\cdot\dot{\underline{v}})^2}{c^2}$$

$$\therefore \quad a_0^2 = \gamma^4 a^2 + \frac{\gamma^6(\underline{v}\cdot\underline{a})^2}{c^2}$$

Motion along straight line $\therefore$ $\underline{v}\cdot\underline{a} = va$.

$$\therefore \quad a_0^2 = \gamma^4 a^2 + \gamma^4\gamma^2\frac{v^2}{c^2}a^2$$

$$= \gamma^4\left(1+\gamma^2\frac{v^2}{c^2}\right)a^2$$

$$= \gamma^4\left(1+\gamma^2\left(1-\frac{1}{\gamma^2}\right)\right)a^2$$

$$= \gamma^4(\cancel{1}+\gamma^2-\cancel{1})a^2 = \gamma^6 a^2.$$

$$\therefore \quad a_0 = \gamma^3 a$$

$\beta = \frac{v}{c}$

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$1-\beta^2 = \frac{1}{\gamma^2}$

$\beta^2 = 1-\frac{1}{\gamma^2}$

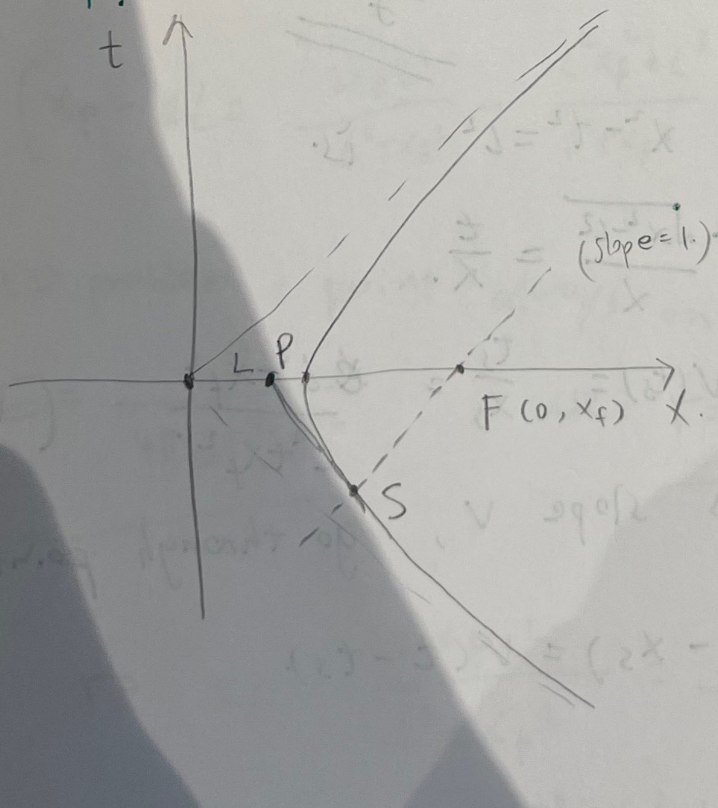
(γ^3 : γ^2 comes from time dilation
γ comes from length contraction)

$$\therefore \quad a_0 = \gamma^3 a = \left(\frac{\cancel{x}}{L}\right)^3 \left(\frac{L^2}{\cancel{x^3}}\right) = \frac{L^2}{L^3} = \frac{1}{L} = \text{const.}$$

- field event : event at which the 4-potential A (thus Electric / magnetic fields) is calculated

- source event : event at which the charge/current generates the fields we are calculating at the field event.

- projected position : the position at which, if the charge moves with constant velocity after the source event, the charge will be in at the time of field event.



F : field event

S : source event

P : projected position

line FS : $t = x - x_f$ (null world line of radiation).

world line $x^2 - t^2 = L^2$

intersection : (t_s, x_s).

$\therefore \quad t_s = x_s - x_f \quad , \quad x_s^2 - t_s^2 = L^2$

$\Rightarrow \quad x_s - t_s = x_f \quad , \quad (\overbrace{x_s - t_s}^{x_f})(x_s + t_s) = L^2$

$\therefore \quad \cancel{x_s + x_f = \frac{L^2}{x}}$ $\begin{cases} x_s + t_s = \dfrac{L^2}{x_f} \\ x_s - t_s = x_f \end{cases}$

$\Rightarrow \quad x_s = \frac{1}{2}\left(\frac{L^2}{x_f} + x_f\right) = \frac{L^2 + x_f^2}{2x_f}$

$t_s = \frac{1}{2}\left(\frac{L^2}{x_f} - x_f\right) = \frac{L^2 - x_f^2}{2x_f}$

tangent line to $x^2 - t^2 = L^2$ is.

$V = \frac{dx}{dt} = - \frac{\sqrt{x^2 - L^2}}{x} = \frac{t}{x}.$

$\therefore$ At $t = t_s$, $v(t_s) = \frac{t_s}{x_s} = \frac{L^2 - x_f^2}{L^2 + x_f^2}.$ $(v < 0)$.

$\therefore$ line SP has slope v, go through point (x_s, t_s)

$\therefore \quad SP : \quad (x - x_s) = v(t - t_s).$

For $t=0$, $x=x_p$

$x_p - x_s = -v t_s$ (v, t_s both <0 $\therefore x < x_s$)

$\therefore \quad x_p = x_s - v t_s$

$$x_p = \frac{L^2+x_f^2}{2x_f} - \frac{L^2-x_f^2}{L^2+x_f^2}\,\frac{L^2-x_f^2}{2x_f}$$

$$= \frac{(L^2+x_f^2)^2-(L^2-x_f^2)^2}{2x_f(L^2+x_f^2)}$$

$$= \frac{(2L^2)(2x_f^2)}{2x_f(L^2+x_f^2)}$$

$$= \frac{\cancel{4}^{2}x_f^{\cancel{2}}L^2}{\cancel{2x_f}(L^2+x_f^2)} = \boxed{\frac{2x_fL^2}{L^2+x_f^2}}$$

$$\left(x_p - L = \frac{2x_fL^2}{L^2+x_f^2} - \frac{L^2x_f^2+L^3}{L^2+x_f^2} = \frac{L(-L^2-x_f^2+2x_fL)}{L^2+x_f^2}\right.$$

$$\left.= \frac{-L(x_f-L)^2}{L^2+x_f^2} \leq 0\right)$$

(1D problem, point always on line of motion)

$$E = \frac{q}{4\pi\varepsilon_0 r^2\gamma^2}$$

use $r = x_f - x_p$

$$r = x_f - \frac{2x_fL^2}{L^2+x_f^2} = \frac{x_f(x_f^2+L^2-2L^2)}{L^2+x_f^2}$$

$$= \frac{x_f(x_f^2-L^2)}{L^2+x_f^2} = \frac{x_f(x_f^2-L^2)}{x_f^2+L^2}$$

$$\gamma = \gamma(x_s) = \frac{x_s}{L} = \frac{L^2 + x_f^2}{2x_f L}$$

$\therefore$

$$E = \frac{q}{4\pi\varepsilon_0} \frac{1}{\left(\frac{x_f(x_f^2 - L^2)}{\cancel{x_f^2 + L^2}}\right)^2} \frac{1}{\left(\frac{\cancel{L^2 + x_f^2}}{2x_f L}\right)^2}$$

$$= \frac{q}{\cancel{4}\pi\varepsilon_0} \frac{(\cancel{2}\cancel{x_f}L)^2}{(\cancel{x_f}(x_f^2 - L^2))^2}$$

$$= \frac{q}{\pi\varepsilon_0} \frac{L^2}{(x_f^2 - L^2)^2}$$

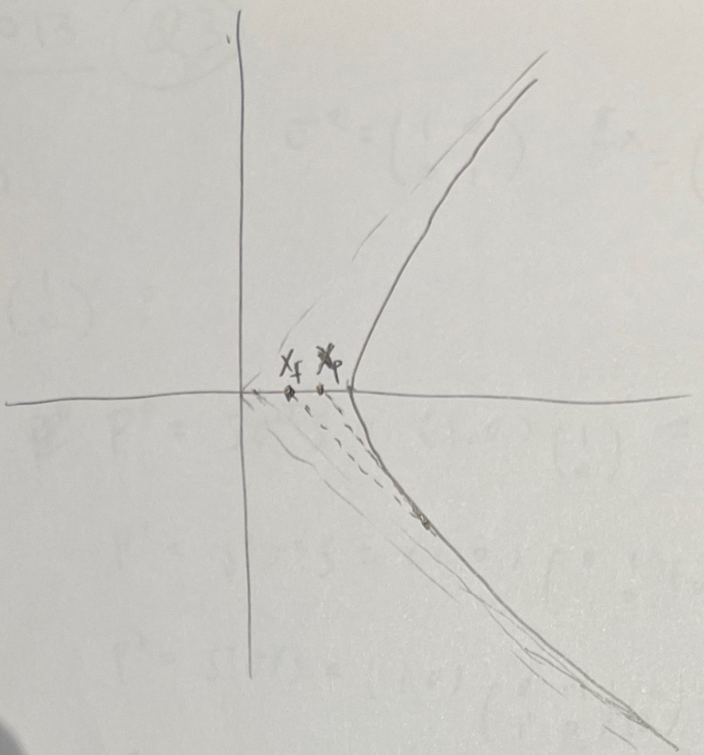
— If $x_f = 2L$ $\quad E = \frac{q}{\pi\varepsilon_0} \frac{L^2}{(4L^2 - L^2)^2} = \frac{q}{\pi\varepsilon_0} \frac{\cancel{L^2}}{9L^{\cancel{4}\,2}}$

$$= \frac{q}{9\pi\varepsilon_0 L^2}$$

(rest at L)

$$E' = \frac{q}{4\pi\varepsilon_0 (2L - L)^2} = \frac{q}{4\pi\varepsilon_0 L^2}$$

$\therefore$ ratio $\frac{E}{E'} = \frac{1/9}{1/4} = \frac{4}{9}$



for $0 < x_f << L$

$$E \simeq \frac{q}{\pi\varepsilon_0}\frac{L^2}{(0-L^2)^2} \simeq \frac{q}{\pi\varepsilon_0 L^2}$$

B2 2013 Q3

(a) $\sigma^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $S = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$P^0 = S^\dagger \sigma^t S = (1, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

$P^1 = S^\dagger \sigma^x S = (1\ 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

$P^2 = S^\dagger \sigma^y S = (1\ 0) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0) \begin{pmatrix} 0 \\ i \end{pmatrix} = 0$

$P^3 = S^\dagger \sigma^z S = (1\ 0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

$\therefore\ P = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

- $S = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. $P^0 = S^\dagger \sigma^t S = \frac{1}{2} (1\ 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \frac{1}{2} (1\ 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$

$P^1 = S^\dagger \sigma^x S = \frac{1}{2} (1\ 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} (1\ 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$

$P^2 = S^\dagger \sigma^y S = \frac{1}{2} (1\ 1) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} (1\ 1) \begin{pmatrix} -i \\ i \end{pmatrix} = 0$

$P^3 = S^\dagger \sigma^z S = \frac{1}{2} (1\ 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} (1\ 1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$

$\therefore\ P = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

- $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ $P^0 = \frac{1}{2} (1\ -i) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}$

$= \frac{1}{2} (1\ -i) \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} (1 - i^2) = \frac{1}{2} (1 - (-1)) = 1$

$$P^1 = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}i\\ 1\end{pmatrix} = 0$$

$$P^2 = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = 1$$

$$P^3 = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1\\ -i\end{pmatrix} = 0$$

$$\therefore P = \begin{pmatrix}1\\ 0\\ 1\\ 0\end{pmatrix}$$

$$- S = \begin{pmatrix}e^{i|\alpha|}\\ 0\end{pmatrix} \qquad P^0 = \begin{pmatrix}e^{-i|\alpha|} & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}e^{i|\alpha|}\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}e^{-i|\alpha|} & 0\end{pmatrix}\begin{pmatrix}e^{i|\alpha|}\\ 0\end{pmatrix} = 1$$

$$P^1 = \begin{pmatrix}e^{-i|\alpha|} & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{i|\alpha|}\\ 0\end{pmatrix} = \begin{pmatrix}e^{-i|\alpha|} & 0\end{pmatrix}\begin{pmatrix}0\\ e^{i|\alpha|}\end{pmatrix} = 0$$

$$P^2 = \begin{pmatrix}e^{-i|\alpha|} & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}e^{i|\alpha|}\\ 0\end{pmatrix} = \begin{pmatrix}e^{-i|\alpha|} & 0\end{pmatrix}\begin{pmatrix}0\\ ie^{i|\alpha|}\end{pmatrix} = 0$$

$$P^3 = \begin{pmatrix}e^{-i|\alpha|} & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{i|\alpha|}\\ 0\end{pmatrix} = \begin{pmatrix}e^{-i|\alpha|} & 0\end{pmatrix}\begin{pmatrix}e^{i|\alpha|}\\ 0\end{pmatrix} = 1$$

$$\therefore P = \begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix}$$

$$\exp(i\sigma^a\theta/2) = I + \frac{i\sigma^a\theta/2}{1!} + \frac{(i\sigma^a\theta/2)^2}{2!} + \frac{(i\sigma^a\theta/2)^3}{3!} + \cdots$$

$$\sigma^a\sigma^a = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I$$

$$\therefore (\sigma^a)^{2n} = I, \quad (\sigma^a)^{2n+1} = \sigma^a \quad \text{for } n \in \mathbb{Z}$$

$$\therefore \exp(i\sigma^a\theta/2) = I\left(1 - \frac{(\theta/2)^2}{2!} + \frac{(\theta/2)^4}{4!} - \cdots\right)$$

$$+ i\sigma^a\left(\theta/2 - \frac{(\theta/2)^3}{3!} + \frac{(\theta/2)^5}{5!} - \cdots\right)$$

$$= \underline{\underline{I\cos\left(\frac{\theta}{2}\right) + i\sigma^a\sin\left(\frac{\theta}{2}\right)}}$$

- rotations $\Lambda = e^{i\sigma\cdot\theta/2}$

start with $S_z = \begin{pmatrix}1\\0\end{pmatrix}$

90° about $x \Rightarrow \Lambda_x = e^{i\sigma_x\frac{\pi}{4}} = I\cos\frac{\pi}{4} + i\sigma_x\sin\left(\frac{\pi}{4}\right)$

$$= \frac{1}{\sqrt{2}}I + i\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\i&1\end{pmatrix}$$

clockwise

$$S' = \Lambda_x S_z = \frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\i&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \underline{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = S_y}$$

counterclockwise

-90° about $y \Rightarrow \Lambda_y = e^{-i\sigma_y\frac{\pi}{4}} = I\cos\frac{\pi}{4},\ i\sigma_y\sin\left(\frac{\pi}{4}\right)$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\+1&1\end{pmatrix}$$

$$\therefore S' = \Lambda_y S_z = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\+1&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \underline{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = S_x}$$

- ~~for~~ to obtain an momentum // $\begin{pmatrix}1\\0\end{pmatrix}$, we. rotate the x-spinor $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ by ~~$\frac{45^\circ}{2} = 22.5^\circ$~~ around $\hat{z}$ by $-\frac{45^\circ}{2} = -22.5^\circ$

by

$$\Lambda_z = e^{-i\sigma_z (22.5^\circ)} = e^{i\sigma_z \frac{\pi}{8}} = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{+i\pi/8} \end{pmatrix}.$$

$$s' = \Lambda_z s_x = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{+i\pi/8} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi/8} \\ e^{+i\pi/8} \end{pmatrix}.$$

Now

$$P^0 = \frac{1}{2} \left(e^{+i\pi/8} \;\; e^{-i\pi/8} \right) \begin{pmatrix} e^{-i\pi/8} \\ e^{+i\pi/8} \end{pmatrix} = 1.$$

$$P^1 = \frac{1}{2} \left(e^{+i\pi/8} \;\; e^{-i\pi/8} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\pi/8} \\ e^{+i\pi/8} \end{pmatrix}.$$

$$= \frac{1}{2} \left(e^{+i\pi/8} \;\; e^{-i\pi/8} \right) \begin{pmatrix} e^{+i\pi/8} \\ e^{-i\pi/8} \end{pmatrix}.$$

$$= \frac{1}{2} \left(e^{-i\pi/4} + e^{i\pi/4} \right) = \cos\left(\frac{\pi}{4}\right). = \frac{1}{\sqrt{2}}.$$

$$P^2 = \frac{1}{2} \left(e^{i\pi/8} \;\; e^{-i\pi/8} \right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-i\pi/8} \\ e^{i\pi/8} \end{pmatrix}.$$

$$= \frac{1}{2} \left(e^{i\pi/8} \;\; e^{-i\pi/8} \right) \begin{pmatrix} -ie^{i\pi/8} \\ ie^{-i\pi/8} \end{pmatrix}$$

$$= \frac{1}{2} \left(-ie^{i\pi/4} + ie^{-i\pi/4} \right) = \frac{1}{2}(-i)(2i)\sin\left(\frac{\pi}{4}\right).$$

$$= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

$$\cancel{P^0}\ P^3 = \frac{1}{2}\begin{pmatrix} e^{+i\pi/8} & e^{-i\pi/8} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\pi/8} \\ e^{i\pi/8} \end{pmatrix}.$$

$$= \frac{1}{2}\begin{pmatrix} e^{+i\pi/8} & e^{-i\pi/8} \end{pmatrix}\begin{pmatrix} e^{-i\pi/8} \\ -e^{i\pi/8} \end{pmatrix} = 0$$

$$\therefore P = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

$\Rightarrow$ spinor is $$S = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\pi/8} \\ e^{i\pi/8} \end{pmatrix}.$$

set phase $$S = \frac{\sqrt{E}}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\pi/4} \end{pmatrix}. =$$

$$\beta = \frac{15}{17} = \tanh\rho \qquad \cosh^2\rho - \sinh^2\rho = 1$$

$$\therefore \cancel{\cosh}\quad \cancel{\tanh\rho}\ \cosh^2\rho - \cosh^2\beta^2 = 1.$$

$$\therefore (\cosh^2\rho) = \frac{1}{1-\beta^2}$$

$$\therefore \gamma = \cosh\rho = \frac{1}{\sqrt{1-\beta^2}} = \frac{17}{8}$$

$$\sinh\rho = \sqrt{\left(\frac{17}{8}\right)^2 - 1} = \gamma\beta = \frac{15}{8}$$

$$e^{-(\rho/2)\sigma_z} = \begin{pmatrix} e^{-\rho/2} & 0 \\ 0 & e^{\rho/2} \end{pmatrix} = \cosh(\rho/2)I - \sinh(\rho/2)\sigma_z$$

$$\cosh^2\left(\frac{\rho}{2}\right) = \frac{1+\cosh\rho}{2} = \frac{25}{16} \Rightarrow \cosh\left(\frac{\rho}{2}\right) = \frac{5}{4}.$$

$$\sinh\left(\frac{\rho}{2}\right) = \sqrt{\left(\frac{5}{4}\right)^2 - 1} = \frac{3}{4}.$$

$$\therefore\; e^{-(\rho/2)\sigma_z} = \frac{5}{4}I - \frac{3}{4}\sigma_z = \begin{pmatrix} 5/4 - \frac{3}{4} & 0 \\ 0 & 5/4 + \frac{3}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow \text{boost}\quad e^{-(\rho/2)\sigma_z}\left(\frac{\sqrt{E}}{\sqrt{2}}\right)\begin{pmatrix} 1 \\ e^{i\pi/4} \end{pmatrix}$$

$$= \frac{\sqrt{E}}{\sqrt{2}}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\pi/4} \end{pmatrix}$$

$$= \frac{\sqrt{E}}{\sqrt{2}}\begin{pmatrix} \frac{1}{2} \\ 2e^{i\pi/4} \end{pmatrix}$$

$$P^0 = \frac{E}{2}\left(\tfrac{1}{2} \;\; 2e^{-i\pi/4}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 2e^{i\pi/4} \end{pmatrix}$$

$$= \frac{E}{2}\left(\frac{1}{4} + 4\right) = \frac{E}{2}\left(\frac{17}{4}\right) = \underline{\underline{\frac{17}{8}E}}$$

$$P^1 = \frac{E}{2}\left(\tfrac{1}{2} \;\; 2e^{-i\pi/4}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 2e^{i\pi/4} \end{pmatrix}$$

$$= \frac{E}{2}\left(\tfrac{1}{2} \;\; 2e^{-i\pi/4}\right)\begin{pmatrix} 2e^{i\pi/4} \\ \frac{1}{2} \end{pmatrix}$$

$$= \frac{E}{2}\left(e^{i\pi/4} + e^{-i\pi/4}\right) = E\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}E$$

$$P^2 = \frac{E}{2}\begin{pmatrix}\frac{1}{2} & 2e^{-i\pi/4}\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2} \\ 2e^{i\pi/4}\end{pmatrix}.$$

$$= \frac{E}{2}\begin{pmatrix}\frac{1}{2} & 2e^{-i\pi/4}\end{pmatrix}\begin{pmatrix}\cancel{-i2e^{i\pi/4}}\ -2ie^{i\pi/4} \\ \frac{1}{2}i\ \cancel{e^{i\pi/4}}\end{pmatrix}.$$

$$= \frac{E}{2}\left(-ie^{i\pi/4} + ie^{-i\pi/4}\right).$$

$$= \frac{E}{2}(\cancel{-i})(\cancel{2i})\sin\left(\frac{\pi}{4}\right). = \cancel{E}\frac{1}{\sqrt{2}}E$$

$$P^3 = \frac{E}{2}\begin{pmatrix}\frac{1}{2} & 2e^{-i\pi/4}\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{2} \\ 2e^{i\pi/4}\end{pmatrix}$$

$$= \frac{E}{2}\begin{pmatrix}\frac{1}{2} & 2e^{-i\pi/4}\end{pmatrix}\begin{pmatrix}\frac{1}{2} \\ -2e^{i\pi/4}\end{pmatrix}.$$

$$= \frac{E}{2}\left(\cancel{2}\frac{1}{4} - 4\right).$$

$$= \frac{E}{2}\left(\frac{-15}{4}\right) = \frac{-15}{8}E$$

$$\therefore\ P' = E\begin{pmatrix}17/8 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ -15/8\end{pmatrix}$$

– the angle ??

Now the vector direction is $\underline{P} = E\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -15/8 \end{pmatrix}$.

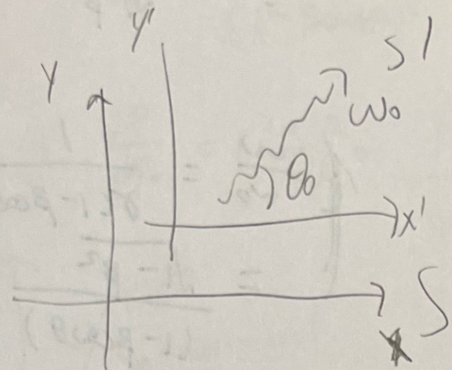
$\underline{P} \cdot \hat{\underline{z}} = |P| \cos\theta$

$$\therefore \cos\theta = \frac{-15/8}{\left((15/8)^2 + (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2\right)^{\frac{1}{2}}} = \frac{-15/8}{17/8} = \frac{-15}{17}.$$

$\therefore \cos\theta = -\frac{15}{17}$ (headlight effect)

$\theta \simeq 152^\circ$

B2 2013 Q4



let $c = \hbar = 1$

consider wave 4-vector

$$K = \begin{pmatrix} \omega \\ \underline{k} \end{pmatrix}$$

photon $\omega = |\underline{k}|$

in S' $\quad K' = \begin{pmatrix} \omega_0 \\ \cancel{\omega_0 \sin\theta_0}\ \omega_0\cos\theta_0 \\ \omega_0 \sin\theta_0 \\ 0 \end{pmatrix}$.

in S $\quad K = \Lambda^{-1} K' = \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_0\cos\theta_0 \\ \omega_0 \sin\theta_0 \\ 0 \end{pmatrix}$.

$$K = \begin{pmatrix} \omega \\ \cancel{\omega}\, k_x \\ k_y \\ 0 \end{pmatrix} = \begin{pmatrix} \omega \\ \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$$

$\therefore \quad \omega = \gamma\omega_0 + \gamma\beta\omega_0\cos\theta_0 \quad ①$

$\omega\cos\theta = \gamma\beta\omega_0 \cancel{\cos\theta_0} + \gamma\omega_0\cos\theta_0 \quad ②$

$\omega\sin\theta = \omega_0\sin\theta_0 \quad ③$

consider 4-vector dot product (invariant under Lorentz tranrformation).

$$K \cdot U = K' \cdot U' \quad (U \text{ is 4 velocity of the source})$$

$$K \cdot U = \begin{pmatrix} \omega \\ \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \gamma c \\ \gamma v \\ 0 \\ 0 \end{pmatrix} = -\gamma\omega c + \gamma\omega v\cos\theta$$

$$K' \cdot U' = \begin{pmatrix} \omega_0 \\ \omega_0\cos\theta_0 \\ \omega_0\sin\theta_0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} = -\omega_0 c$$

$\therefore \quad \omega_0 c = \gamma \omega c \left(1 - \frac{v}{c}\cos\theta\right)$ $\quad \because \gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$\omega_0 = \gamma \omega (1 - \beta\cos\theta)$$

$$\therefore \quad \omega = \frac{\omega_0}{\gamma(1-\beta\cos\theta)}$$

$$\left(\frac{\omega}{\omega_0} = \frac{1}{\gamma(1-\beta\cos\theta)} = \frac{\sqrt{1-\beta^2}}{(1-\beta\cos\theta)} \right)$$

consider ①, ②

$\Rightarrow \quad \frac{\cos\theta \text{ ①}}{\gamma\omega_0} \Rightarrow \quad \frac{\omega\cos\theta}{\gamma\omega_0} = \cos\theta(1 + \beta\cos\theta_0)$

$\frac{\text{②}}{\gamma\omega_0} \Rightarrow \quad \frac{\omega\cos\theta}{\gamma\omega_0} = \cos\theta_0 + \beta$

$$\therefore \quad \cos\theta + \beta\cos\theta\cos\theta_0 = \cos\theta_0 + \beta$$

$$\therefore \quad \cos\theta_0(1 - \beta\cos\theta) = \cos\theta - \beta$$

$$\Rightarrow \quad \cos\theta_0 = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}$$

— power per solid angle :

$d\Omega = \sin\theta \, d\theta \, d\phi$, $\quad d\Omega_0 = \sin\theta_0 \, d\theta_0 \, d\phi_0$

$d\phi = d\phi_0$ (no effect in transverse direction)

$\sin\theta \, d\theta = -d(\cos\theta) \quad \sin\theta_0 \, d\theta_0 = -d(\cos\theta_0)$.

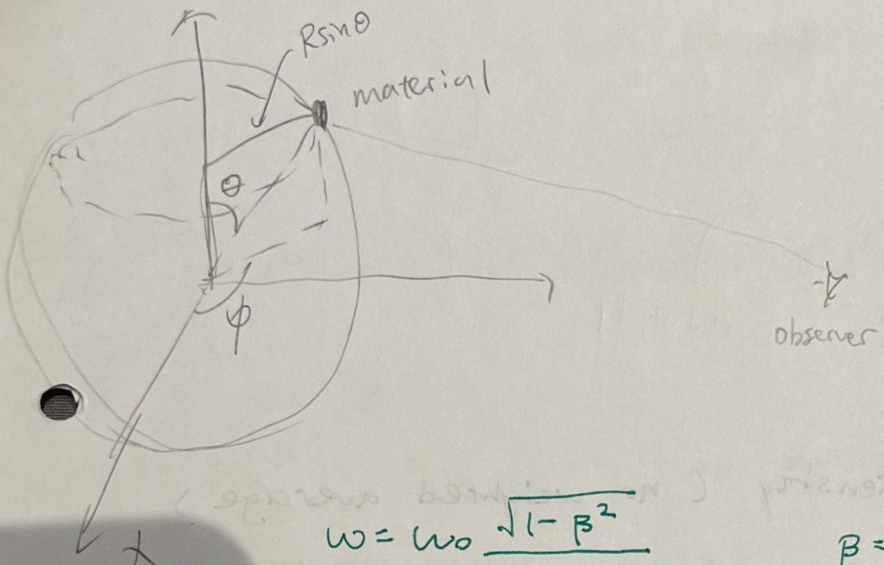
$\therefore \quad \frac{d\Omega_0}{d\Omega} = \frac{-d(\cos\theta_0)\cancel{d\phi_0}}{-d(\cos\theta)\cancel{d\phi}} = \frac{d\cos\theta_0}{d\cos\theta} = \frac{df_0}{df} \quad \left(\begin{array}{l} f_0 \equiv \cos\theta_0 \\ f \equiv \cos\theta \end{array} \right)$

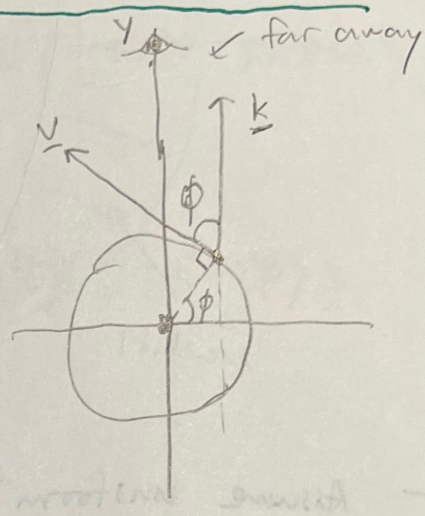
$f_0 = \frac{f - \beta}{1 - \beta f} \quad \therefore \quad \frac{df_0}{df} = \frac{(1-\beta f)(1) - (f-\beta)(-\beta)}{(1-\beta f)^2} = \frac{1 - \beta f + \beta f - \beta^2}{(1-\beta f)^2}$

$$= \frac{(1-\beta^2)}{(1-\beta f)^2}$$

$$\therefore \quad \frac{d\Omega_0}{d\Omega} = \frac{1-\beta^2}{(1-\beta\cos\theta)^2} \qquad \because \text{ intensity } \frac{I}{I_0} = \left(\frac{\omega}{\omega_0}\right)^2 = \frac{dP}{dP_0}$$

$$\therefore \quad \frac{dP}{dP_0} = \left(\frac{\omega}{\omega_0}\right)^2 = \frac{1-\beta^2}{(1-\beta\cos\theta)^2} \Rightarrow \frac{dP}{d\Omega} = \frac{(1-\beta^2)^2}{(1-\beta\cos\theta)^4}\frac{dP_0}{d\Omega_0}$$





$$\omega = \omega_0 \frac{\sqrt{1-\beta^2}}{(1-\beta\cos\Phi)} \qquad \beta = \Omega R\sin\theta = \Omega\sqrt{R^2-z^2}$$

Φ : angle between velocity of material and direction of emission of radiation in observer's rest frame S.

By simple geometry $\Phi = \phi$ $\qquad \beta\cos\Phi = \beta\cos\phi = \Omega R\sin\theta\cos\theta$

$= \Omega x$

$$\therefore \quad \omega = \omega_0 \frac{\sqrt{1-\Omega^2(R^2-z^2)}}{1-\Omega x}$$

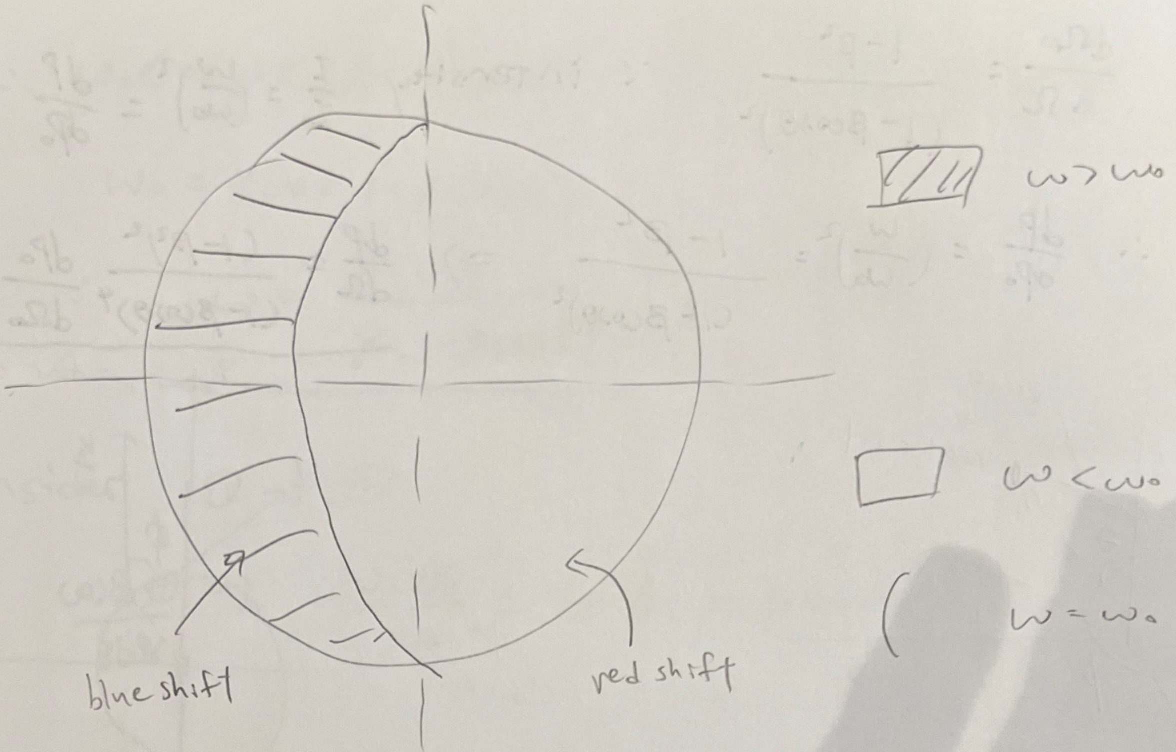
- if $\omega = \omega_0$, $\quad 1-\Omega x_c = \sqrt{1-\Omega^2(R^2-z^2)}$

$$\therefore \quad x_c = \frac{1}{\Omega}\left(1-\sqrt{1-\Omega^2(R^2-z^2)}\right) > 0$$

when $x > x_c$, $\quad \omega > \omega_0$

Hence

$x < x_c$, $\quad \omega < \omega_0$



- Assume uniform intensity (no weighted average)

$$\lambda\omega = 2\pi c \quad \therefore \ \omega\, d\lambda + \lambda\, d\omega = 0 \quad \therefore \ \frac{d\lambda}{\lambda} + \frac{d\omega}{\omega} = 0$$

$$\therefore \ \frac{\Delta\lambda}{\lambda_0} \simeq -\frac{\Delta\omega}{\omega_0} = \frac{\omega_0 - \omega}{\omega_0} = 1 - \frac{\omega}{\omega_0}$$

$$= 1 - \frac{\sqrt{1-\Omega^2(R^2-z^2)}}{1-\Omega x} = 1 - \frac{\sqrt{1-\Omega^2R^2\sin^2\theta}}{1-\Omega R\sin\theta\cos\phi}$$

$$\Omega R = \frac{\Omega R}{c} = \frac{2\times10^6\times10^3\,\cancel{m}\times3\times10^{-4}\,\cancel{s^{-1}}}{3\times10^8\,\cancel{m\,s^{-1}}} = \frac{6\times10^5}{3\times10^8} = \frac{1}{500} \ll 1$$

$$\therefore \ \frac{\Delta\lambda}{\lambda_0} \simeq 1 - \left(1 - \frac{1}{2}\Omega^2R^2\sin^2\theta - \frac{1}{8}\Omega^4R^4\sin^4\theta\right)$$

$$\times\left(1 + \Omega R\sin\theta\cos\phi + \Omega^2R^2\sin^2\theta\cos^2\phi + \Omega^3R^3\sin^3\theta\cos^3\phi + \Omega^4R^4\sin^4\theta\cos^4\phi + \cdots\right)$$

$$= 1 - \Big(1 - \frac{1}{2}\Omega^2R^2\sin^2\theta + \Omega R\sin\theta\cos\phi + \Omega^2R^2\sin^2\theta\cos^2\phi$$

$$+ \Omega^3R^3\sin^3\theta\cos^3\phi - \frac{1}{2}\Omega^3R^3\sin^2\theta\cos\theta\cos\phi$$

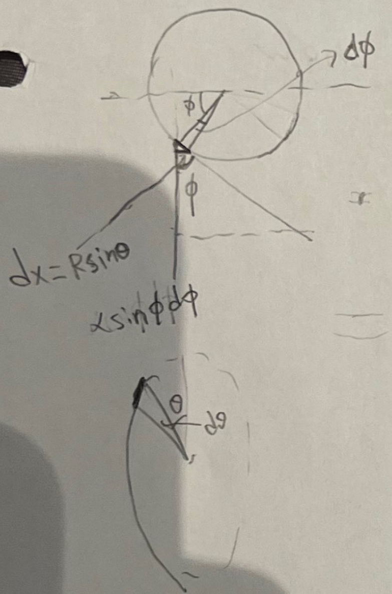
$$- \frac{1}{8}\Omega^4R^4\sin^4\theta + \Omega^4R^4\sin^4\theta\cos^4\phi + O(R^5\Omega^5)\Big).$$

$$\therefore \frac{\Delta\lambda}{\lambda_0} \simeq 1 - \left(1 - \frac{1}{2}\Omega^2 R^2 \sin^2\theta + O(\Omega^4 R^4) + \cdots\right) \times$$

$$\left(1 + \Omega R \sin\theta \cos\phi + \Omega^2 R^2 \sin^2\theta \cos^2\phi + O(\Omega^3 R^3) + \cdots\right)$$

$$= \cancel{1} - \cancel{1} - \Omega R \sin\theta\cos\phi + \Omega^2 R^2\left(\frac{1}{2}\sin^2\theta - \sin^2\theta\cos^2\phi\right) + O(\Omega^3 R^3)$$

$$\therefore \frac{\langle\Delta\lambda\rangle}{\lambda_0} = -\Omega R\langle\sin\theta\rangle\langle\cos\phi\rangle + \Omega^2 R^2 \langle\sin^2\theta\rangle\left(\frac{1}{2} - \langle\cos^2\phi\rangle\right)$$



$\langle\ \rangle$ is average

~~$\langle f(\theta)\rangle$ in sp~~

$$\langle f(\phi)\rangle = \frac{\int_0^\pi f(\phi)\sin\phi\, d\phi}{\int_0^\pi \sin\phi\, d\phi}$$

$$\langle f(\theta)\rangle = \frac{\int_0^\pi f(\theta)\sin\theta\, d\theta}{\int_0^\pi \sin\theta\, d\theta}$$

$$\Rightarrow \langle\sin^2\theta\rangle = \frac{2}{3}, \quad \langle\cos^2\phi\rangle = \frac{1}{3} \quad \cancel{\langle\sin} \ \langle\cos\phi\rangle = 0$$

$$\therefore \frac{\Delta\lambda}{\lambda_0} = \Omega^2 R^2 \left(\frac{2}{3}\right)\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2}{3} \times \frac{1}{6} \times \Omega^2 R^2$$

$$= \frac{\Omega^2 R^2}{9}$$

$$\therefore \Delta\lambda \simeq 500 \times \frac{1}{9} \times \frac{1}{500^2} = \underline{\underline{2 \times 10^{-4}\, \text{nm}}}$$

$$\cancel{= 2 \times 10^{-13}\, \text{m}} \qquad \sim \underline{\underline{10^{-13}\, \text{m}}}$$

<u>redshifted.</u>

- ~~non~~ infact brightness is not uniform

$\because$ $\frac{dP}{d\Omega} = \left(\frac{\omega}{\omega_0}\right)^4 \frac{dP}{d\Omega_0}$.

$\therefore$ the higher frequency side of the disk is brighter

→ we need weighted average

high frequency has larger weight

→ disk, on average, will now become ~~blushi~~ blue shifted