BPhO 2024 Round 2

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Solutions

1 Question 1

(a) The Earth rotates at an angular speed of

$$\Omega = \frac{2\pi}{24 \times 60 \times 60} = 7.27 \times 10^{-5} \text{ rad/s} \tag{1}$$

If the plane moves with the same angular speed relative to the axis of the Earth and at the opposite direction, then relative to the Sun it does not move (within one day period at least), so the radius of the circle that the plane moves is

$$r = \frac{v}{\Omega} = \frac{900}{3.6 \times 7.27 \times 10^{-5}} = 3.44 \times 10^{6} \text{ m} \tag{2}$$

And the latitude ϕ is given by

$$\cos\phi = \frac{r}{R_E} \tag{3}$$

where R_E is the radius of the Earth, which gives $\phi = 57.4°$.

(b) Consider two droplets of water A and B initially adjacent in the bucket. At time $t = 0$ the droplet A starts free fall, and after a small time interval $t = \delta t$ droplet B starts free fall. After time t_1, droplet A lands on the ground. So the height of the tower is given by

$$h = \frac{1}{2}gt_1^2 \tag{4}$$

The initial distance between A and B when B just starts free fall is

$$\Delta h_1 = \frac{1}{2}g(\delta t)^2 \tag{5}$$

The distance that B has travelled when A just landed on the ground is

$$h' = \frac{1}{2}g(t_1 - \delta t)^2 = \frac{1}{2}gt_1^2 + \frac{1}{2}g(\delta t)^2 - gt_1\delta t \tag{6}$$

So at the moment when A lands on the ground, the distance between A and B is

$$\Delta h_2 = h - h' = gt_1\delta t - \frac{1}{2}g(\delta t)^2 \tag{7}$$

Initially the droplets are very close to each other in the bucket so $\delta t \ll t_1$, so

$$\frac{\Delta h_2}{\Delta h_1} = \frac{2t_1}{\delta t} - 1 \gg 1 \tag{8}$$

which gives $\Delta h_2 \gg \Delta h_1$ which means the final separation between droplets will be much larger than the initial separation, hence in the form of rain.

Of course, in real world physics there is air resistance which will disturb the free fall, but this can still cause the stream of water to break up into smaller sections because the air exerts drag forces on different parts of the water stream. The broken pieces of water then form droplets to minimise surface area under the effect of surface tension.

(c)

(i) Using $Q = CV$, it is easy to see that initially both capacitors are charged with 72 μC, say positive on the left and negative on the right plates. Conservation of charge dictates that after connection the left plates of two capacitors sum to 144 μC and right plates sum to -144 μC. Also after connection the potential across the two capacitors should be the same, let this potential difference be V we have

$$C_1 V + C_2 V = 144 \tag{9}$$

which gives

$$V = \frac{144}{(6+12)} = 8 \text{ V} \tag{10}$$

which is the potential difference across both C_1 and C_2.

(ii) The initial energy

$$U = \frac{1}{2}(6)(12)^2 + \frac{1}{2}(12)(6)^2 = 648 \;\; \mu\text{J} \tag{11}$$

The final energy

$$U' = \frac{1}{2}(6)(8)^2 + \frac{1}{2}(12)(8)^2 = 576 \;\; \mu\text{J} \tag{12}$$

So the fraction dissipated is

$$\frac{648 - 576}{648} = \frac{1}{9} \tag{13}$$

(d) This is a very special stable solution of the three body problem. The three equal masses form a equilateral triangle and moves together in a circular orbit around the centre of the triangle as shown in the figure.

The force between any pair of masses is

$$F' = \frac{Gm^2}{(\sqrt{3}r)^2} = \frac{Gm^2}{3r^2} \tag{14}$$

The centripetal force is the vectorial sum of the two F's which has the magnitude

$$F = \sqrt{3}F' = \frac{\sqrt{3}Gm^2}{3r^2} \tag{15}$$

Using the equation for circular motion gives

$$\frac{\sqrt{3}Gm^2}{3r^2} = F = m\omega^2 r \tag{16}$$

which gives the angular speed

$$\omega = \left(\frac{1}{3}\right)^{\frac{1}{4}} \frac{\sqrt{Gm}}{r^{3/2}} \tag{17}$$

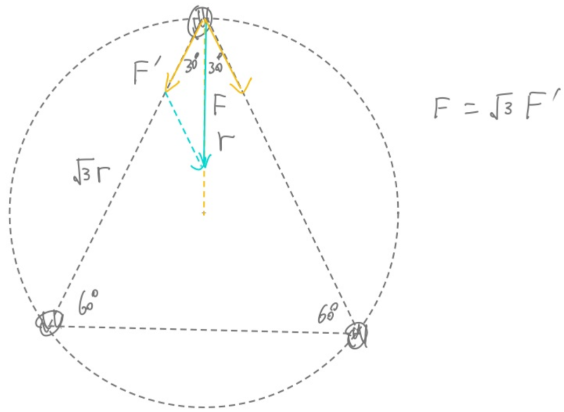

Figure 1: Orbit of 3 masses

So period is

$$T = \frac{2\pi}{\omega} = \frac{(3^{1/4})2\pi}{\sqrt{Gm}} r^{3/2} \tag{18}$$

(e) We take

$$I(T) = I_0 \mathrm{e}^{-E_g/k_B T} \tag{19}$$

Substituting in values gives

$$\frac{10.90}{3.97} = \mathrm{e}^{-\frac{E_g}{k_B}\left(\frac{1}{90+273} - \frac{1}{60+273}\right)} \tag{20}$$

which gives $E_g = 5.62 \times 10^{-20}$ J $= 0.35$ eV.

For very large T the current is I_0, we have

$$I_0 = 10.90 \times e^{E_g/k_B(90+273)} = 8.1 \times 10^5 \text{ A} \tag{21}$$

2 Question 2

(a) According to the Bernoulli's principle, the air travels through the upper side of the wing has higher speed and thus lower pressure, so the difference in pressures and thus in forces on the two sides of the wing lifts the wing and thus the aircraft upwards.

(b) For $\alpha = 15°$, from the charts we have $C_L = 1.43$ and $C_D = 0.027$. Shortly after take off we ignore the drag in the vertical direction. The vertical lift force is given by

$$L = \frac{1}{2}C_L\rho A v^2 = \frac{1}{2}(1.43)(1.3)(845)(90)^2 = 6.36 \times 10^6 \ \text{N} \tag{22}$$

and the vertical acceleration is

$$a_V = \frac{L - mg}{m} = \frac{L}{m} - g = \frac{6.36 \times 10^6}{5.6 \times 10^5} - 9.8 = 1.56 \ \text{m/s}^2 \tag{23}$$

Passengers experience this vertical acceleration through the normal force from the seat.

(c) Constant altitude implies the drag is completely horizontal. For $\alpha = 8°$, from the charts we have $C_L = 0.90$ and $C_D = 0.012$.

Resolving vertically we have the lift equals the weight

$$mg = L = \frac{1}{2}C_L\rho A v^2 \tag{24}$$

which gives

$$v = \sqrt{\frac{2mg}{C_L\rho A}} = 134.3 \ \text{m/s} \tag{25}$$

Resolving horizontally we have the engine thrust equals the drag so

$$F_T = D = \frac{1}{2}C_D\rho A v^2 = 7.32 \times 10^4 \ \text{N} \tag{26}$$

and the useful power is

$$P = F_T v = 9.83 \times 10^6 \ \text{W} \tag{27}$$

(d) Again since we have steady and level flight,

$$mg = \frac{1}{2}C_L\rho A v^2 \tag{28}$$

which gives

$$C_L = \frac{2mg}{\rho A v^2} = 0.58 \tag{29}$$

which correspond to a angle of attack

$$\alpha = 5.2° \tag{30}$$

on the figure. This same angle corresponds to $C_D = 0.009$, so the thrust

$$F_T = \frac{1}{2}C_D\rho A v^2 = 8.56 \times 10^4 \ \text{N} \tag{31}$$

and thus the useful power

$$P = F_T v = 2.14 \times 10^7 \ \text{W} \tag{32}$$

(e) When α is small, lift coefficient is very small so to fly steady and level we need a very large speed, which increases the drag. When α is very large, the drag coefficient becomes very large so this increases the drag as well.

(f) In a steady and level flight, using

$$mg = \frac{1}{2}C_L \rho A v^2 \tag{33}$$

and

$$D = \frac{1}{2}C_D \rho A v^2 \tag{34}$$

we can divide and get

$$D = \frac{C_D}{C_L} mg \tag{35}$$

Hence to minimise D we just need to minimise the ratio between coefficients $\frac{C_D}{C_L}$.

Looking at the figures we find that this ratio is minimum at angle of attack between 9 and 10 degrees, so we estimate

$$\alpha = 9.5^\circ \tag{36}$$

3 Question 3

(a) Only the component of the optical field parallel to the transmission axis of the polarisation filter (polariser) will pass through the device. Hence when two mutually perpendicular polarisers are placed one after the other, the optical field that passes the first polariser will have no component parallel to the second one, and thus to light transmitted throught the second polariser (*Remark: look for Malus's Law).

(b) When an electromagnetic wave enters a dielectric medium, it excites (resonates) the material's electrons whether they are free or bound, setting them into a vibratory state with the same frequency as the wave. By Ewald–Oseen extinction theorem, the overall wave inside the material has speed $v = \frac{c}{n}$ and the same frequency as the original wave in vacuum.

Since frequency does not change, the wavelength inside the material becomes $\lambda' = \frac{\lambda}{n}$, so for a total width of t, the number of wavelengths is

$$M = \frac{t}{\lambda'} = \frac{tn}{\lambda} \tag{37}$$

as required.

(c) Since one wavelength is a phase difference of 2π, the slower wave has traveled

$$M_s = \frac{tn_s}{\lambda} \tag{38}$$

number of wavelengths and the faster wave has traveled

$$M_f = \frac{tn_f}{\lambda} \tag{39}$$

number of wavelengths. So if they are initially in phase, and knowing that the $n_s > n_f$ because $v = \frac{c}{n}$, in total the phase difference is

$$\Delta\phi = 2\pi M_s - 2\pi M_f = \frac{2\pi t}{\lambda}(n_s - n_f) = \frac{2\pi}{\lambda} t\Delta n \tag{40}$$

as required

(d) To maximise I we need $\sin^2(2\alpha) = 1$, this gives $\alpha = 45°$ and $\alpha = 135°$

When the fast axis is aligned with the axis of the first filter, there is no slow wave gets transmitted through the first filter. Hence there is no phase difference, and thus no rotation of the plane of polarisation. The second filter will complete block the fast wave as if there is no birefringent material. Similar logic works if the slow axis is aligned with the first filter.

(e) We have

$$\begin{aligned} I &= \frac{I_0}{2}\sin^2(2\alpha)\sin^2(\frac{\Delta\phi}{2}) \\ &= \frac{I_0}{2}\sin^2(2\alpha)\sin^2(\frac{\pi}{\lambda}t\Delta n) \end{aligned} \tag{41}$$

For $\alpha = 45°$, $\Delta n = 0.011$ and $t = 30\ \mu$m, we have

$$I = \frac{I_0}{2}\sin^2(\frac{1036.7}{\lambda}) \tag{42}$$

where λ is in unit of nanometers (nm) and the argument of sine function is in radians.

(f) For this new thickness the intensity function becomes

$$I = \frac{I_0}{2}\sin^2(\frac{3110.2}{\lambda}) \tag{43}$$

The sketch for both (e) and (f) are as shown, where the purple curve is for (e) and the blue

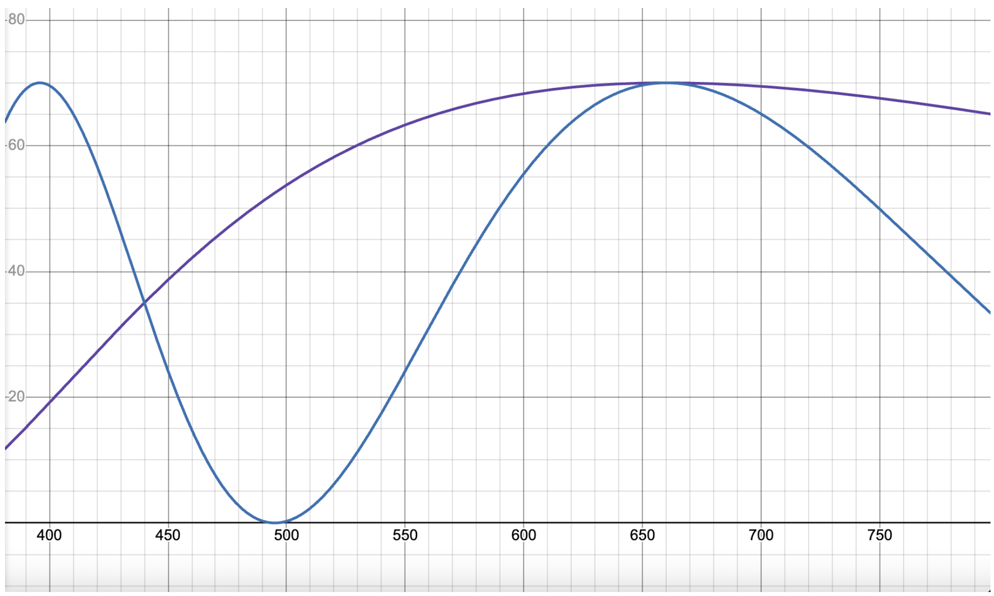

Figure 2: Q3 (e) (f)

curve is for (f), and the horizontal axis is wavelength in nm.

The apparent color for the material in (e) is red (maximum at 650nm), and that for the material in (f) is magenta which is a combination of red (650nm) and violet (400nm).

(g) Now the transmitted intensity function becomes

$$I = I' \sin^2(\frac{10000\pi}{510}\Delta n) \tag{44}$$

where $I' = \frac{I_0}{2}\sin^2(2\alpha)$ and $\lambda = 510$ nm is the wavelength for green light. The 10000 nm is the thickness.

For maximum intensity transmittted we need the sine function to be ± 1 so that

$$\frac{10000\pi}{510}\Delta n = \frac{2m-1}{2}\pi \tag{45}$$

for $m = 1, 2, 3, \cdots$, which gives

$$\Delta n = 0.051 \times \frac{2m-1}{2} \tag{46}$$

We observe a stark green color with no red or blue, this means that within the $\lambda \in (400, 780)$ nm range the intensity function needs to have no minima (so the intensity won't rise again at the blue or red ends) but at the same time as sharply peaked as possible. The best choice is when $m = 2$ and we have

$$\Delta n = 0.0765 \tag{47}$$

(h) When thickness t becomes very large, the transmitted intensity I in the $\lambda \in (400, 780)$ nm range varies extremely rapidly with λ. Hence very closely spaced maxima of intensity essentially appear at every wavelength within the visible range, which makes the overall light white.

(i) During the production of polymers like cellotape and cellophane, the material is often stretched or extruded. This stretching causes the long-chain polymer molecules to become aligned in a specific direction. As a result, the material exhibits anisotropy and hence its optical properties vary with direction.

(j) Cellophane film were stacked several layers thick and measured with calipers.

(k) For calcite with appropriate Δn, look at the transmission function

$$I_t(\lambda) = I' \sin^2(\frac{0.172 \cdot \pi}{\lambda} t) \tag{48}$$

where $I' = \frac{I_0}{2} \sin^2(2\alpha)$ is a constant. For a given t this function has vary rapid oscillations at small λ and oscillation starts to have longer and longer period in λ as λ gets large (peaks have larger and larger width), until when λ gets sufficiently large where there is the last oscillation after which the function just decays to 0. When we increase the thickness t this function keeps the same shape but expands horizontally, so the same shape of oscillation/peak now corresponds to a larger wavelength.

When t is too small, the $\lambda \in (400, 780)$ nm range is located at the rightmost decaying part of this function where transmitted intensity is too small to be observed. As we increase t this range will begin to contain the last (rightmost) peak, then the second last peak, then the third last peak etc. This analysis is how we solve (g) as well.

To estimate the minimum thickness needed to observe colourful birefringent effects, we set the right part of the rightmost peak of the transmission function where the value of $I_t(\lambda)$ equals half of the peak value I' to be at $\lambda = 400$ nm, the shortest wavelength of visible range. This means we would like to find the smallest t such that

$$\sin(\frac{0.172 \cdot \pi}{400} t) = \frac{1}{\sqrt{2}} \tag{49}$$

which gives

$$t \approx 0.6 \ \ \mu\text{m} = 6 \times 10^{-7} \ \ \text{m} \tag{50}$$

as our estimate.

Now, the mass of a single Calcite molecule is given by the molar mass divided by the Avogadro's Number,

$$m = \frac{M}{N_A} = \frac{100 \times 10^{-3}}{6.022 \times 10^{23}} = 1.66 \times 10^{-25} \ \ \text{kg} \tag{51}$$

So the average volume occupied by a single molecule is

$$V = \frac{\rho}{m} = \frac{1.66 \times 10^{-25}}{2.71 \times 10^3} = 6.13 \times 10^{-29} \ \ \text{m}^3 \tag{52}$$

So average distance between two molecules is

$$l = V^{1/3} = 3.94 \times 10^{-10} \ \ \text{m} \tag{53}$$

and the number of layers is

$$\frac{6 \times 10^{-7}}{3.94 \times 10^{-10}} \approx 1500 \tag{54}$$

4 Question 4

(a)

(i) Energy of a single photon is given by

$$E = h\nu = \frac{hc}{\lambda} \tag{55}$$

where ν is the frequency and λ is the wavelength. So energy is inversely proportional to the wavelength.

The typical wavelength of a UV photon is 300 nm, and a typical wavelength of a visible light photon is 600 nm, so the efficiency is approximately

$$\frac{300}{600} = 50\% \tag{56}$$

This result is necessarily a maximum because some energy is always lost as heat during the process, and not every absorbed UV photon will necessarily result in a visible photon.

(ii) It can be seen that the efficiency of fluorescent light bulbs is higher than that of incandescent light bulbs.

Fluorescent light bulbs use an electric discharge to excite mercury vapor, which emits ultraviolet (UV) light. The UV light then excites a phosphor coating on the inside of the bulb, causing it to emit visible light. This two-step process is more efficient at converting electrical energy into visible light. Incandescent light bulbs produce light by heating a tungsten filament until it glows. A lot of the electrical energy is lost as heat, as only a small portion is converted into visible light. Fluorescent bulbs operate at lower temperatures, which minimizes energy lost as heat and improves overall efficiency.

(b)

(i) Rearrange the previous equation we can get

$$\lambda = \frac{hc}{E} \tag{57}$$

for a photon, where $E = 1.42$ eV is the energy between the band gap. The emission from the de-excitation of GaAs thus has wavelength $\lambda = 875$ nm. This is not near the range around 494 nm and this means that GaAs cannot stimulate fluorescence in fluorescein.

(ii) The laser is the collimated light. In time Δt, number of photons emitted from the source is $\frac{P_l \Delta t}{E_g}$ since the band gap is the energy of emitted photon. Multiplying this with ε gives number of photons that fluorescein absorbs, then further multiplying with Φ gives the number of photon re-emitted by the fluorescein isotopically, which is $\varepsilon\Phi\frac{P_l \Delta t}{E_g}$. Isotropy implies spherical symmetry which shows that the proportion of $\frac{A}{4\pi r^2}$ of re-emitted photon arrives at the detector which is then amplified by a factor of f. The total number of photon counted at the detector is $\Delta n = f\frac{A}{4\pi r^2}\varepsilon\Phi\frac{P_l \Delta t}{E_g}$. Hence

$$\frac{\Delta n}{\Delta t} = \frac{f\varepsilon\Phi A P_l}{4\pi r^2 E_g} \tag{58}$$

(iii) In this particular set-up, we have $P_l = 4\times10^{-3}$ W, $\Phi = 0.9$, $\varepsilon = 0.01$, $A = \frac{\pi}{4}(0.01)^2 = 7.85\times10^{-5}\ \mathrm{m}^2$, $f = 10^6$, and $E_g = 2.50\ \mathrm{eV} = 4.0\times10^{-19}$ J. The current $I = 3\times10^{-3}$ A is given by

$$I = \frac{\Delta Q}{\Delta t} = \frac{e\Delta n}{\Delta t} = \frac{ef\varepsilon\Phi AP_l}{4\pi r^2 E_g} \tag{59}$$

which gives

$$r = \sqrt{\frac{ef\varepsilon\Phi AP_l}{4\pi IE_g}} = 0.173\ \mathrm{m} \tag{60}$$

as the distance between the sample and the detector. In reality not all photon re-emitted from fluorescein in the initial direction towards the detector can arrive at the detector. It may be absorbed by some other impurities along the way, thus reducing the estimation of $\varepsilon\Phi P_l$. Also the re-emission may not be perfectly isotropic, and in most cases the detector is not placed in the direction of the initial laser so the effective receiving proportion of area is less than $\frac{A}{4\pi r^2}$. These factors may be the cause of an overestimation of r.

(iv) We know that $r \propto \sqrt{A}$ and $A \propto d^2$, so $r \propto d$. Say $r = kd$, then differentiate gives $\delta r = k\delta d$. Dividing these two equations gives

$$\frac{\delta d}{d} = \frac{\delta r}{r} = 5\% \tag{61}$$

(c)

(i) This mathematical relation can be expressed as

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN \tag{62}$$

which can be solve to

$$N(t) = N_0\mathrm{e}^{-kt} \tag{63}$$

where N_0 is the population of excited fluorophores at $t = 0$.

Since

$$\frac{N_0}{\mathrm{e}} = N_0\mathrm{e}^{-k\tau} \tag{64}$$

we have $-1 = -k\tau$ and hence

$$\tau = \frac{1}{k} \tag{65}$$

It is obvious that $[\tau] = \mathrm{s}$ and $[k] = \mathrm{s}^{-1}$.

(ii) The current is proportional to the number of photons hitting the detector per unit time which is proportional to the number of fluorophores present excited. After the pulse laser the number of fluorophores decays according to the equation above. So

$$0.25 = 3\,\mathrm{e}^{-(10\,\mathrm{ns})/\tau} \tag{66}$$

which gives

$$\tau = \frac{10}{\ln\frac{3}{0.25}} = 4.02\ \mathrm{ns} \tag{67}$$