BPhO 2023 Round 2 Question 4

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Space Elevator - Solutions

(a) The centrifugal force equals to the gravitational force

$$m_s\omega^2 r_g = \frac{Gm_e m_s}{r_g^2} \tag{1}$$

and since $\omega = \frac{2\pi}{T}$, we have

$$T = \frac{2\pi}{\sqrt{Gm_e}} r_g^{3/2} \tag{2}$$

For geostationary satellite $T = 1$ day. Hence

$$r_g = \left(\frac{Gm_e}{4\pi^2}\right)^{1/3} T^{2/3} = 4.22 \times 10^7\,\mathrm{m} \tag{3}$$

(b) Let this extra force be F, we have

$$m_s\omega^2(r_g + r') + F = \frac{Gm_e m_s}{(r_g + r')^2} = \frac{Gm_e m_s}{r_g^2}(1 + \frac{r'}{r_g})^{-2} \tag{4}$$

so by binomial expansion

$$F = \frac{Gm_e m_s}{r_g^2}(1 - \frac{2r'}{r_g}) - m_s\omega^2 r_g - m_s\omega^2 r' \tag{5}$$

since $m_s\omega^2 r_g = \frac{Gm_e m_s}{r_g^2}$, we have

$$\omega^2 r_g^3 = Gm_e \tag{6}$$

and hence

$$F = -m_s\omega^2 r' - 2m_s\omega^2 r' = -3m_s\omega^2 r' \tag{7}$$

with positive direction defined to be directed radially outwards from the Earth towards space, which is a force

$$|F| = 3m_s\omega^2 r' \tag{8}$$

radially inwards if $r' > 0$.

(c) We integrate to get the total gravitational force on the elevator

$$F_{ge} = \int_{r_e}^{r_g} \frac{Gm_e\mu \mathrm{d}r}{r^2} = Gm_e\mu(\frac{1}{r_e} - \frac{1}{r_g}) \tag{9}$$

and integrate the total centrifugal force on the elevator

$$F_{ce} = \int_{r_e}^{r_g} \omega^2 r \mu \mathrm{d}r = \frac{1}{2}\omega^2(r_g^2 - r_e^2)\mu \tag{10}$$

Now look at the mass m. Since m is at position $r_g + r'$ from Earth, from the results of previous questions there is a tension force

$$|F| = 3m\omega^2 r' \tag{11}$$

radially inwards holding this mass. Hence this same tension force must be pulling the elevator radially outwards in the same direction of the centrifugal force. Hence we have $F_{ce} + |F| = F_{ge}$ and that

$$\frac{1}{2}\omega^2(r_g^2 - r_e^2)\mu + 3m\omega^2 r' = Gm_e\mu(\frac{1}{r_e} - \frac{1}{r_g}) \tag{12}$$

Using $\omega^2 r_g^3 = Gm_e$ we have

$$\omega^2 r_g^3(\frac{1}{r_e} - \frac{1}{r_g})\mu = \frac{1}{2}\omega^2(r_g^2 - r_e^2)\mu + 3m\omega^2 r' \tag{13}$$

Eliminating ω gives

$$\frac{r_g^3}{r_e} - r_g^2 = \frac{1}{2}r_g^2 - \frac{1}{2}r_e^2 + \frac{3r'm}{\mu} \tag{14}$$

Using $r_e \ll r_g$ (the ratio is about 10% , which is just about OK), we can only keep the first term on the LHS and the last term on the RHS to get

$$\frac{r_g^3}{r_e} = \frac{3r'm}{\mu} \tag{15}$$

which gives

$$m = \frac{\mu r_g^3}{3r_e r'} \tag{16}$$

as required.

(d) A typical string has mass per unit length $\mu \approx 10^{-4}$ kg/m, and assume $r' = 1000$ m, we have $m = 3.93 \times 10^8$ kg which is too much to be realistic.

(e) Analysing the balance of forces on a small piece dr on the cable. We have mass per unit length $\mu = \rho A$ and

$$\sigma(r + \mathrm{d}r)A + \omega^2 r\rho A\mathrm{d}r = \sigma(r)A + \frac{Gm_e}{r^2}\rho A\mathrm{d}r \tag{17}$$

which represents the centrifugal force plus tensile force outwards equals the gravitational force plus tensile force inwards. Note that $\mathrm{d}\sigma = \sigma(r + \mathrm{d}r) - \sigma(r)$ and simplify gives

$$\mathrm{d}\sigma = (\omega^2 r - \frac{Gm_e}{r^2})\rho\mathrm{d}r \tag{18}$$

Again using $\omega^2 r_g^3 = Gm_e$ we have $\omega^2 r = \frac{Gm_e r}{r_g^3}$ and that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}r} = Gm_e\rho\left(\frac{1}{r^2} - \frac{r}{r_g^3}\right) \tag{19}$$

as required.

(f) The largest tensile stress occurs obviously at where $\frac{\mathrm{d}\sigma}{\mathrm{d}r} = 0$ which is precisely at $r = r_g$. To find the tensile stress we integrate the above result such that

$$\sigma(r) = \int_{r_e}^{r} Gm_e\rho\left(\frac{1}{r''^2} - \frac{r''}{r_g^3}\right)\mathrm{d}r'' = Gm_e\rho\left(\frac{1}{r_e} - \frac{1}{r} - \frac{r^2 - r_e^2}{2r_g^3}\right) \tag{20}$$

Hence when $r = r_g$ we have maximum tensile stress

$$\sigma(r_g) = Gm_e\rho\left(\frac{1}{r_e} - \frac{3}{2r_g} + \frac{r_e^2}{2r_g^3}\right) = (4.84 \times 10^7)\rho \tag{21}$$

in SI units.

For steel, this is 382 GPa; for Kevlar, this is 67.8 GPa; and for Carbon nanotubes this is 62.9 GPa. Only Carbon nanotubes can stand this stress.

For later use we also want to calculate the total height H of this elevator. At the ends there is no stress, so we need $\sigma(r) = 0$, which gives us the equation

$$\frac{1}{r_e} - \frac{1}{r} - \frac{r^2 - r_e^2}{2r_g^3} = 0 \tag{22}$$

which can be reduced to

$$2r_g^3(r - r_e) = r_e r(r + r_e)(r - r_e) \tag{23}$$

we observe that $r = r_e$ is definitely a solution, and this of course represents the lower end, which is not the solution we want and we can safely divide the term $r - r_e$ to give us

$$r^2 + r_e r - \frac{2r_g^3}{r_e} = 0 \tag{24}$$

which can be easily seen gives one positive and one negative roots. We want the positive root so

$$r_+ = \frac{1}{2}\left(\sqrt{r_e^2 + \frac{8r_g^3}{r_e}} - r_e\right) \approx \sqrt{\frac{2r_g^3}{r_e}} = 1.5 \times 10^8\,\mathrm{m} \tag{25}$$

And the height of the elevator H is calculated by

$$H = r_+ - r_e \approx 1.5 \times 10^8\,\mathrm{m} \tag{26}$$

as we can see subtracting the radius of the Earth does not make a difference to the final result for the significant figures we are using.

(g) Using the exact the same force balance analysis as previous questions, keeping in mind the stress σ is now a constant, we have

$$\sigma A(r + \mathrm{d}r) + \omega^2 r\rho A(r)\mathrm{d}r = \sigma A(r) + \frac{Gm_e}{r^2}\rho A(r)\mathrm{d}r \tag{27}$$

Rearrange and again use $\omega^2 r_g^3 = Gm_e$ gives

$$\sigma\frac{\mathrm{d}A}{A} + \frac{Gm_e}{r_g^3}\rho r\mathrm{d}r = Gm_e\rho\frac{\mathrm{d}r}{r^2} \tag{28}$$

and hence using $A(r_e) = A_b$,

$$\int_{A_b}^{A(r)} \frac{\mathrm{d}A'}{A'} = \frac{Gm_e\rho}{\sigma} \int_{r_e}^{r} \left(\frac{1}{r''^2} - \frac{r''}{r_g^3} \right) \mathrm{d}r'' \tag{29}$$

so

$$\ln \frac{A(r)}{A_b} = \left(\frac{Gm_e}{r_e^2} \right) \frac{r_e^2 \rho}{\sigma} \left[\frac{1}{r_e} - \frac{1}{r} - \frac{r^2 - r_e^2}{2r_g^3} \right] \tag{30}$$

Note that $g = \frac{Gm_e}{r_e^2}$ is the gravitational acceleration on the surface of the Earth. Given that $L_c = \frac{\sigma}{\rho g}$. Integrate both sides we have

$$A(r) = A_b \exp \left[\frac{r_e^2}{L_c} \left(-\frac{1}{r} - \frac{r^2}{2r_g^3} + k \right) \right] \tag{31}$$

as required, where from previous calculations it is clear that

$$k = \frac{1}{r_e} + \frac{r_e^2}{2r_g^3} \tag{32}$$

Now we go back to the stage before separating the variables and performing integrations. It is easy to get

$$\frac{\sigma}{A} \frac{\mathrm{d}A}{\mathrm{d}r} = Gm_e\rho \left(\frac{1}{r^2} - \frac{r}{r_g^3} \right) \tag{33}$$

For maximum area we need $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$, so this obviously happens at $r = r_g$ again. And the maximum area is

$$A(r_g) = A_b \exp \left[\frac{r_e^2}{L_c} \left(\frac{1}{r_e} - \frac{3}{2r_g} + \frac{r_e^2}{2r_g^3} \right) \right] \tag{34}$$

(h) At $r = r_e$, $A(r_e) = A_b$. Hence

$$\frac{A(r_g)}{A_b} = \exp \left[\frac{r_e^2}{L_c} \left(\frac{1}{r_e} - \frac{3}{2r_g} + \frac{r_e^2}{2r_g^3} \right) \right] = \exp \left[\frac{r_e}{2L_c} (s^3 - 3s + 2) \right] \tag{35}$$

where $s = \frac{r_e}{r_g} = 0.151$, and substituting in numerical values gives

$$\frac{A(r_g)}{A_b} = \mathrm{e}^{\frac{4.944 \times 10^6}{L_c}} \tag{36}$$

We assume maximum tensile stress. For carbon nanotubes, we assume $\sigma = 130$ GPa, so $L_c = 1.02 \times 10^7$ m, which gives $\frac{A(r_g)}{A_b} = 1.624$. For steel we have $\sigma = 5$ GPa, so $L_c = 6.46 \times 10^4$ m, which gives $\frac{A(r_g)}{A_b} = 1.73 \times 10^{33}$. For Kevlar we have $\sigma = 3.6$ GPa, so $L_c = 2.62 \times 10^5$ m, which gives $\frac{A(r_g)}{A_b} = 1.57 \times 10^8$. It is clear from these numbers that only carbon nanotubes give reasonable answers.

(i) For symmetrical construction the area of cross sections at the two ends are equal. So we have $A(r_t) = A_b$ where, r_t is the distance from the top of the elevator to the centre of the Earth. So we have

$$-\frac{1}{r_t} - \frac{r_t^2}{2r_g^3} + k = -\frac{1}{r_t} - \frac{r_t^2}{2r_g^3} + \frac{1}{r_e} + \frac{r_e^2}{2r_g^3} = 0 \tag{37}$$

which, after rearrangement, gives

$$2r_g^3(r_t - r_e) = r_e r_t (r_t + r_e)(r_t - r_e) \tag{38}$$

which is exactly the same equation as we encountered in the calculations in question (f). So knowing that $r_t \neq r_e$ we of course have the same solution

$$r_t = \frac{1}{2}\left(\sqrt{r_e^2 + \frac{8r_g^3}{r_e}} - r_e\right) \approx \sqrt{\frac{2r_g^3}{r_e}} = 1.5 \times 10^8\,\mathrm{m} \tag{39}$$

And the height of elevator $H' = r_t - r_e = r_+ - r_e = H$ is the same as the uniform area elevator in question (e).

Remark: This question is an adaptation (and of course a much simpler version) of *APhO 2018 Theory Question 2*. In David Morin's book *Introduction to Classical Mechanics: with Problems and Solutions*, the Exercise 5.65 also provides a great account for the theory of a uniform area space elevator.

But, as stated much more clearly in the original APhO question, the uniform stress space elevator actually needs a counterweight of an appropriate mass, because if the top end is free, the stress at the top end would turn suddenly from σ to 0, which is impossible as in this case the very top differential volume $A\mathrm{d}r$ won't balance according to equation (27) because the $\sigma A(r + \mathrm{d}r)$ term doesn't exist. The APhO question actually asked contestants to calculate the mass of this counterweight, but this BPhO question just completely ignored its existence.