BPhO

2022

Round 2

Solutions

Ziyan Li

① (a) $P = Fv \Rightarrow F = \frac{P}{V}$

force of propulsion

$\therefore \Sigma F = ma = \frac{P}{V} - \rho A V^2 \geq 0$

for maximum $V \Rightarrow \frac{P}{V} = k\rho A V^2$

$$\Rightarrow k = \frac{P}{\rho A V^3}$$

$V = 240$ ml/hr $= 107.3$ m/s

$P = 461 \times 10^3$ W

$\rho = 1.29$ kg/m³

$A =$ Area of car $\simeq 1.8\text{m} \times 1.5\text{m} \simeq 2.7\text{m}^2$

$$\Rightarrow \underline{k = 0.107}$$

(b)

$V(t) = V_0 \sin(2\pi f t)$

$I(t) = \frac{V(t)}{R} = \frac{V_0}{R}\sin(2\pi f t)$

$\Rightarrow P(t) = V(t) I(t) = \frac{V_0^2}{R}\sin^2(2\pi f t)$

period $T = \frac{1}{f}$

Average Power $\bar{P} = \frac{1}{T}\int_0^T P(t)dt = \frac{V_0^2 f}{R}\int_0^{\frac{1}{f}} \sin^2(2\pi f t)\,dt$

Note that $\cos 2\theta = 1 - 2\sin^2\theta$

$\Rightarrow \quad \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

$\sin(4\pi f t)$

$$\Rightarrow \quad \bar{P} = \frac{V_0^2 f}{R}\int_0^{\frac{1}{f}} \frac{1}{2} - \frac{1}{2}\cos(4\pi f t)\,dt$$

$$= \frac{V_0^2 f}{R}\left(\left(\frac{1}{2}\right)\left[\frac{1}{f} - 0\right] - \frac{1}{8\pi f}\sin(4\pi f t)\Big|_0^{\frac{1}{f}} \right)$$

$$= \frac{V_0^2}{2R}$$

total time last $t_0 = \frac{E}{\bar{P}} = \frac{2ER}{V_0^2}$

(C) larger particles → larger gaps/spaces between particles

→ smaller particles can fall through gaps/spaces

(d) - the front window is more slanted compare to the sides so the front has more exposure to the cold sky

BPhO 2022 Round 2 Question 2

Ziyan Li

Stellar Interiors - Solutions

(a) The repulsive force between two protons is

$$F_e = \frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{(1.6\times10^{-19})^2}{4(3.14)(8.85\times10^{-12})(7\times10^{-15})^2} = 4.7\,\mathrm{N} \tag{1}$$

which is very close to 5 N as required.

(b) As shown in the figure, the upper purple curve is the electrostatic Coulomb force, the lower

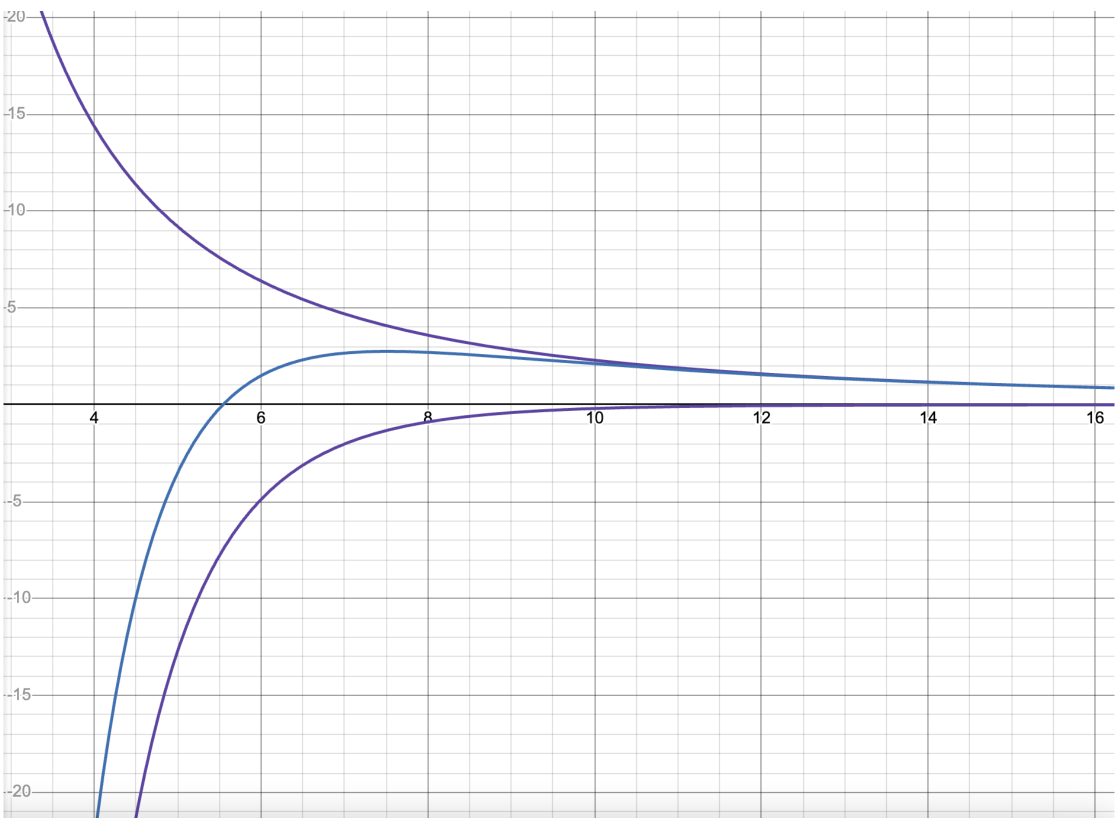

Figure 1: Forces between two protons

purple curve is the strong nuclear force (more precisely the residual effect of the strong nuclear force, governed by Yukawa coupling), and the blue curve is the resultant force.

High temperature is required for fusion to occur because there is a Coulomb barrier to prevent the protons from approaching each other, which will be very obvious if the above force curves are integrated to the potential energy curves.

We assume the ideal gas law and use the average kinetic energy per particle as the kinetic energy of a single proton

$$K = \frac{3}{2} k_B T \tag{2}$$

The Coulomb barrier has radius $r_c \approx 5.5$ fm and a potential energy of

$$U = \frac{e^2}{4\pi\varepsilon_0 r_c} \tag{3}$$

We need the kinetic energy of two protons to overcome the Coulomb barrier so $2K > U$, this gives

$$T \geq \frac{e^2}{12\pi\varepsilon_0 k_B r_c} = 10^9\,\mathrm{K} \tag{4}$$

(c) This is likely to be an overestimate for various reasons. First, the attractive strong nuclear force at the radius of the Coulomb barrier can help the protons cross the barrier. Second, the protons do not all travel at the average velocity. Assume a Maxwell-Boltzmann distribution there is a tail of a very small fraction of particles that can travel at much higher speeds at the same temperature. And last but not least, the tunnelling effect in quantum mechanics allows the protons to pass the potential well of the Coulomb barrier even without enough speed calculated by classical physics.

(d) Since $g(r)$ has a minus sign we know the positive direction of gravitational force is defined to be outwards. Balancing the forces acting on the parcel of fluid gives

$$p(r + \mathrm{d}r)A = \rho(r)g(r)A\mathrm{d}r + p(r)A \tag{5}$$

, which by rearrangement gives

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \rho(r)g(r) \tag{6}$$

This is less than 0 because $\rho(r)$ is positive and $g(r)$ is negative, so pressure decreases with increasing radius.

(e) We crudely estimate that the density of the Sun is constant such that

$$\rho(r) = \rho_0 = \frac{M}{\frac{4}{3}\pi R^3} = 1.4 \times 10^3\,\mathrm{kg/m^3} \tag{7}$$

hence the mass $m(r)$ is

$$m(r) = \rho_0 \frac{4}{3}\pi r^3 = \frac{Mr^3}{R^3} \tag{8}$$

so

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho_0 \frac{Gm(r)}{r^2} = -\frac{M}{\frac{4}{3}\pi R^3} \cdot \frac{G}{r^2} \cdot \frac{Mr^3}{R^3} = -\frac{3GM^2}{4\pi R^6} r \tag{9}$$

It is safe to assume that at the surface of the sun the pressure is 0, so this gives the integration constant and we have

$$p(r) = \int_R^r -\frac{3GM^2}{4\pi R^6} r' \mathrm{d}r' = \frac{3GM^2}{8\pi R^6}(R^2 - r^2) \tag{10}$$

At the centre we have

$$p_c = p(0) = \frac{3GM^2}{8\pi R^4} = 1.33 \times 10^{14}\,\mathrm{Pa} \tag{11}$$

Use ideal gas results

$$pV = Nk_BT \tag{12}$$

For a spherical shell of volume from r to $r + \mathrm{d}r$, this can be applied to give

$$p(r)\mathrm{d}V = (\mathrm{d}N)k_BT(r) = \frac{\rho_0 \mathrm{d}V}{m_p}k_BT(r) \tag{13}$$

where m_p is the proton mass, so at the centre we have

$$T_c = \frac{m_p p_c}{\rho_0 k_B} = 1.15 \times 10^7\,\mathrm{K} \tag{14}$$

Since the previous estimate about the lowest temperature for fusion to occur is certainly an overestimate, it is plausible that the Sun is powered by fusion. The speed of a single proton at the core is

$$\frac{1}{2}m_p v^2 = \frac{3}{2}k_BT_v \tag{15}$$

so

$$v = \sqrt{\frac{3k_BT_c}{m_p}} = 5.34 \times 10^5\,\mathrm{m/s} \approx 0.002c \tag{16}$$

so the proton is not moving relativistically.

(f) Using the given density profile, we have

$$m(r) = \int_0^r 4\pi r'^2 \rho_0 (1 - \frac{r'^2}{R^2})\mathrm{d}r' = 4\pi\rho_0(\frac{r^3}{3} - \frac{r^5}{5R^2}) \tag{17}$$

This also means that

$$M = m(R) = \frac{8}{15}\pi\rho_0 R^3 \tag{18}$$

so

$$\rho_0 = \frac{15M}{8\pi R^3} = 3.48 \times 10^3\,\mathrm{kg/m^3} \tag{19}$$

The pressure gradient is given by

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho_0(1 - \frac{r^2}{R^2})\frac{G}{r^2}4\pi\rho_0(\frac{r^3}{3} - \frac{r^5}{5R^2}) \tag{20}$$

so

$$p(r) = \int_R^r 4\pi\rho_0^2 G(1 - \frac{r'^2}{R^2})(\frac{r'}{3} - \frac{r'^3}{5R^2})\mathrm{d}r' \tag{21}$$

hence

$$p(r) = \frac{2\pi G\rho_0^2(2R^2 - r^2)(R^2 - r^2)^2}{15R^4} \tag{22}$$

and that

$$p_c = p(0) = \frac{4}{15}\pi G\rho_0^2 R^2 = 3.32 \times 10^{14}\,\mathrm{Pa} \tag{23}$$

and similarly because ρ_0 is precisely the density at the centre

$$T_c = \frac{m_p p_c}{\rho_0 k_B} = 1.15 \times 10^7\,\mathrm{K} \tag{24}$$

(g) For a gas cloud to be gravitationally bounded and able to collapse to form a star, the total energy, which includes the thermal kinetic energy and gravitational potential energy, must be less than zero. This inequality provides a critical lower bound for the mass of the star that can be formed. This is known as the Jeans mass (Jeans criterion).

On the other hand, after the mass of the star has reached a certain upper limit, its mass should have become hot enough for its heat to drive away any further incoming matter. It reaches a point where it evaporates away material already collected as fast as it collects new material. This upper limit is known as the accretion limit.

So in general the mass of a star is bounded both below and above.

Remark: The solutions presented in this article contain very crude approximations. To be less crude, when calculating the temperature profile, we actually need to use the adiabatic relation between pressure and volume and then use the virial theorem to find the relationship between the energy density and pressure. We also need to take into account the heat transfer by photon diffusion so the temperature gradient inside the star can be properly calculated when it is related to the luminosity. When force caused by radiation can balance the gravitational pull, another explanation to the the upper limit of the mass of the star arises, which is known as the Eddington limit. More details can be found in Chapter 35 of the book by Stephan and Katherine Blundell *Concepts in Thermal Physics*.

BPhO 2022 Round 2 Question 3

Ziyan Li

The Runner's Pace Function - Solutions

(a) Increasing m means increasing the component of gravity that acts down the slope of the hill thus making it harder to run the same pace.

(b) On flat ground the speed

$$v_0 = \frac{1}{p(0)} = \frac{1000}{3.5 \times 60} = 4.76\,\mathrm{m/s} \tag{1}$$

(c) The pace function can be written as

$$p(m) = 0.035(m+3)^2 + 3.185 \tag{2}$$

so the pace is minimum, which also means the speed is maximum, at $m = -3$ and the corresponding pace is

$$p_{\min} = p(-3) = 3.185\,\mathrm{min/km} \tag{3}$$

(d) The pace function is sketched as shown This pace function would not be valid for all m

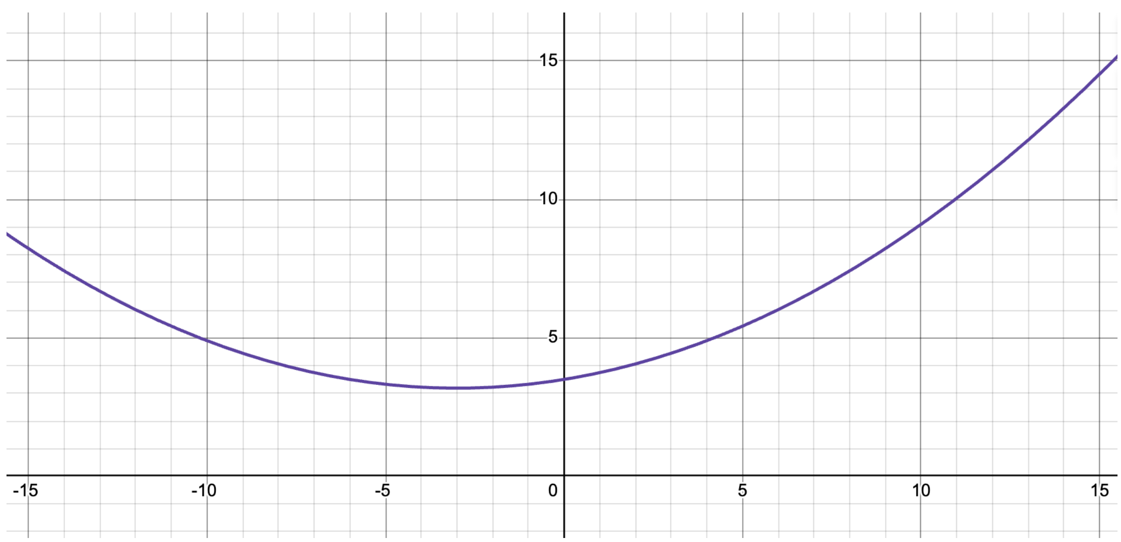

Figure 1: the pace function

because as m becomes more and more negative, at some finite value of m the person would fall down and start to roll down the hill instead of running with longer and longer pace. And as positive m gets larger and larger there will be a finite value of m such that the person can't run uphill anymore and p goes to ∞.

part	distance d (km)	gradient m	p (min/km)	v (m/s)	t (s)
0-3.5 (km)	3.5	2	4.06	4.11	851.6
3.5-5 (km)	1.5	-2	3.22	5.18	289.6
5-9 (km)	4	0.5	3.61	4.62	865.8
9-10 (km)	1	-6	3.50	4.76	210.1
overall	10	N/A	N/A	N/A	2217.1

Table 1:

(e) We break the route into 4 parts as shown in the table, the total time taken is $t = 2217.1$ s, which is equivalent to $t = 37.0$ mins. So the average pace is

$$\bar{p} = \frac{37}{10} = 3.7\,\text{min/km} \tag{4}$$

The result in minutes can also be calculated directly using $t = \int_0^d p\,\mathrm{d}x = \sum_i p_i d_i$.

(f) The pace is defined to be

$$p = \frac{\mathrm{d}t}{\mathrm{d}x} \tag{5}$$

and the gradient is defined to be

$$m = 100 \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \tag{6}$$

Therefore the rate to gain vertical height, r, is given by

$$r = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{m}{100p} = \frac{m}{100(3.5 + 0.21m + 0.035m^2)} \tag{7}$$

r can also be re-written as

$$r(m) = \frac{1}{3.5(\frac{100}{m} + m) + 21} \tag{8}$$

Consider

$$(a^2 + b^2) - 2ab = (a - b)^2 \geq 0 \tag{9}$$

which implies

$$a^2 + b^2 \geq 2ab \tag{10}$$

and the equality holds when $a = b$.

Now let $a = \frac{10}{\sqrt{m}}$ and $b = \sqrt{m}$ we have

$$\frac{100}{m} + m \geq 2 \cdot \frac{10}{\sqrt{m}} \cdot \sqrt{m} = 20 \tag{11}$$

and the equality holds when $m = 10$. And we observe that when $\frac{100}{m} + m$ achieves minimum, $r(m)$ achieves maximum.

So the maximum rate of gaining height is

$$r_{\text{max}} = r(10) = 0.011\,\text{km/min} = 0.183\,\text{m/s} \tag{12}$$

and of course the critical gradient is $m = 10$.

(g) For a linear pace function

$$p(m) = a + bm \tag{13}$$

We re-write this using $p = \frac{dt}{dx}$ and $m = \frac{dy}{dx}$ such that

$$dt = a\,dx + b\,dy \tag{14}$$

We are using algebraic symbols instead of numbers here, so no percentage to ratio conversion factor of 100 here for m. Note that x is not the x-coordinate or a displacement here. It has nothing to do with vectors or a vector component. This x is purely the scalar distance travelled in the horizontal plane. But the y in this equation has vectorial nature and can be positive or negative depending on running uphill or downhill. So if the total horizontal distance travelled is d, and using the fact that since the runner starts and finishes at the same place, her initial and final y coordinate should be the same y_0, we have

$$\int_0^t dt' = \int_0^d a\,dx + \int_{y_0}^{y_0} b\,dy \tag{15}$$

and the y integral vanishes and

$$t = ad = p(0)d \tag{16}$$

independent of b. Hence the total run time is independent of the elevation profile.

For the critical gradient, since

$$r = \frac{dy}{dt} = \frac{m}{p} = \frac{m}{a + bm} = \frac{1}{\frac{a}{m} + b} \tag{17}$$

We want to find the maximum rate. Observe that when $m \longrightarrow -\frac{a}{b}$ from the positive side r approaches $+\infty$ and when from the negative side r approaches $-\infty$, which does not make any physical sense. The pace function cannot be valid for human near $m = -\frac{a}{b}$.

If we want to find a local maximum, that is, a point where $\frac{dr}{dm} = 0$, we have

$$\frac{dp}{dm} = \frac{p}{m} \tag{18}$$

and that gives $\frac{a}{m} = 0$, which also does not make any physical sense.

The critical gradient does not exist in this case.

BPhO 2022 Round 2 Question 4

Ziyan Li

Cloud Physics - Solutions

(a)

(i) Initially, the pressure inside the flask increases because the water molecules near the liquid surface can go through the process of evaporation. As more and more water evaporates into water vapor, the pressure above the liquid surface increases. At the same time, a fraction of the molecules in the vapor phase will collide with the surface of the liquid and re-enter the liquid phase through the process of condensation. The rate of condensation increases as more vapor has formed. When the rate of condensation becomes the same as the rate of evaporation, dynamic equilibrium is reached and no more vapor forms. Hence the pressure eventually levels off.

(ii) Using the ideal gas relation $pV = Nk_BT$, and the fact that $N = nV$, where n is the number density of the water vapor, thus we have

$$p = nk_BT \tag{1}$$

As T decreases we see that p should also decrease. However as T decreases the rate of condensation will become more than the rate of evaporation so overall there will be some vapor turns into liquid water. As this happens, because obviously liquid water is much denser than water vapor, n for water vapor will decrease, therefore both n and T on the right hand side decrease, so p will decrease. And since n decreases, the $p(T)$ curve will not be perfectly linear, instead the slope becomes less and less steep as T decreases.

The $p - T$ curve looks like the following.

(b) As shown in the figure

(c) Separating variables and integrate gives

$$\int_{p_0}^{p_{\text{sat}}} \frac{\mathrm{d}p}{p} = \frac{LM}{R}\int_{T_0}^{T} \frac{\mathrm{d}T'}{T'^2} \tag{2}$$

which gives the required expression

$$\ln\left(\frac{p_{\text{sat}}}{p_0}\right) = \frac{LM}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right) \tag{3}$$

(d) The molar mass of water H_2O is given by $2(1) + 1(16) = 18$ g/mol. Using $p_0 = 0.006$ atm and $T_0 = 0.01 + 273.15 = 273.16$ K and $T = 20 + 273.15 = 293.15$ K and the given values of L and R, we have

$$p_{\text{sat}} = 2340\,\text{Pa} \tag{4}$$

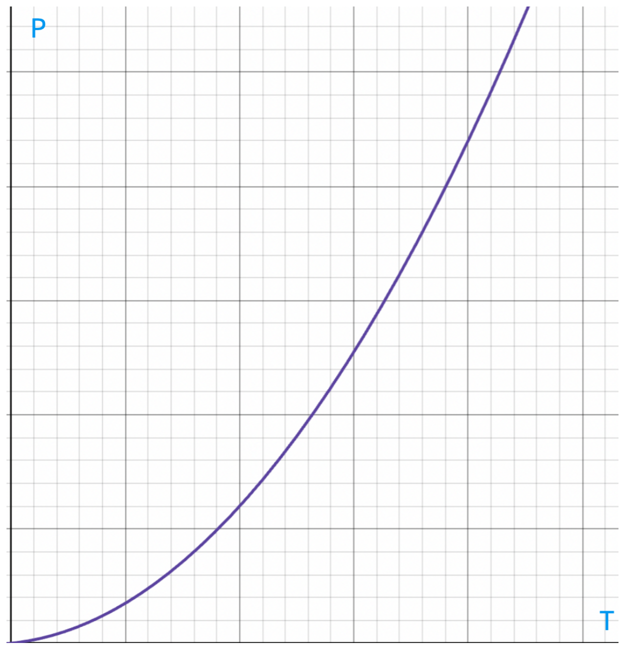

Figure 1: question a (ii), the $p - T$ curve

(e) Warm air rises due to lower density, and as the temperature cools at higher altitude the saturation pressure decreases, so the relative humidity increases. Convective clouds form in the lower atmosphere when warm, humid air near the ground rises due to convection, cools as it ascends, and reaches a point where the relative humidity reaches 100% and saturation occurs, causing water vapor to condense into visible cloud droplets; essentially, the higher the relative humidity near the surface, the lower the altitude at which convective clouds can form due to less cooling needed to reach saturation.

(f) At 20 degrees, the partial pressure is given by

$$p_{\mathrm{w}} = \phi \cdot p_{\mathrm{sat}} = 35\% \times 2340 = 819\,\mathrm{Pa} \tag{5}$$

Use $T = 293.15$ K

$$p_{\mathrm{w}} = n_{\mathrm{w}} k_B T \tag{6}$$

we obtain $n_{\mathrm{w}} = 2.2 \times 10^{23}$ number of water molecules per $1\,\mathrm{m}^3$.

(g) At sea level, atmospheric pressure is $p_a = 1\,\mathrm{atm} = 1.01 \times 10^5$ Pa. By Dalton's Law, partial pressure is proportional to the number density, so the number density of air n_a is given by

$$\frac{n_{\mathrm{w}}}{n_a} = \frac{819}{1.01 \times 10^5} \tag{7}$$

which gives $n_a = 2.7 \times 10^{25}$ air molecules per $1\,\mathrm{m}^3$.

Assume by fraction of number of molecules air consists of 79% of Nitrogen, which has number density n_n and 21% of Oxygen, which has number density of n_o. We have $n_n = 2.133 \times 10^{25} = 97.0 n_{\mathrm{w}}$ and $n_o = 5.67 \times 10^{24} = 25.77 n_{\mathrm{w}}$.

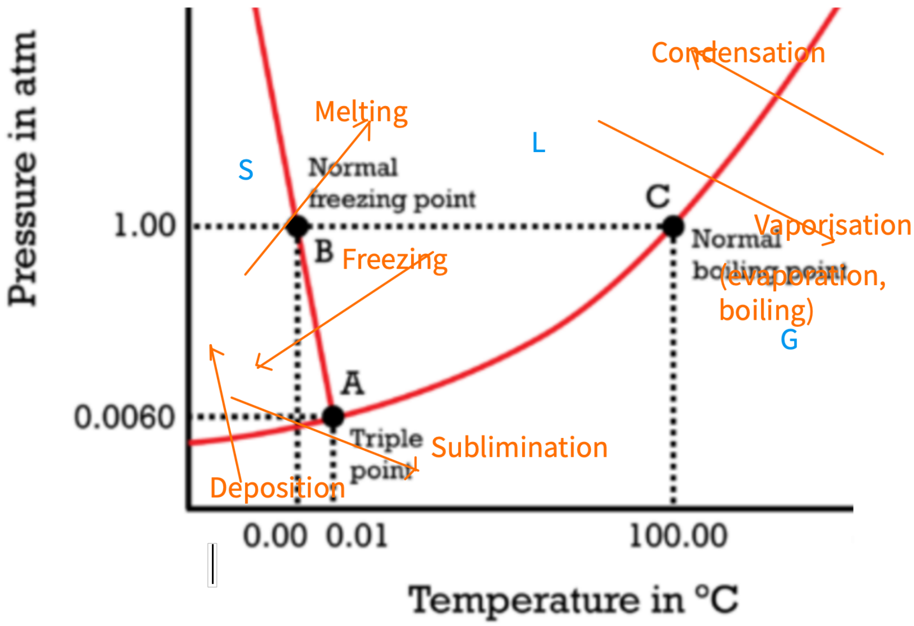

Figure 2: question (b)

Since the mass density is proportional to the molar mass and the number density, we have the mass fraction of water equals to approximately

$$\frac{18 \cdot 1}{18 \cdot 1 + 28 \cdot 97 + 16 \cdot 25.77} = 0.5\% \tag{8}$$

(h) Cloud formation occurs when the partial pressure of water vapor equals to the saturation pressure. As the warm humid air rises and cools, the fraction in number density of water vapor molecules in that of air molecules remains unchanged. So we always have

$$p_{\mathrm{w}}(T) = \frac{819}{1.01 \times 10^5} \cdot p_a = 0.008 p_a(T) \tag{9}$$

As the air rises, the saturation pressure is given by the equation derived in (c). We have $\frac{LM}{R} = 5415.16$ and

$$p_0 \mathrm{e}^{\frac{LM}{RT_0}} = 2.466 \times 10^{11} \tag{10}$$

so

$$p_{\mathrm{sat}}(T) = (2.466 \times 10^{11}) \mathrm{e}^{-\frac{5415.16}{T}} \tag{11}$$

where T is in Kelvins. We then have the following result outlined in table (1).

We then take the following very rough estimates. We estimate that for every 750 m up in altitude, the partial pressure of water vapor drops around 70 Pa, and the temperature goes down around 10 K. So at an altitude of around 2250 m, We have $p_{\mathrm{w}} = 610$ Pa, and $p_{\mathrm{sat}} = 451$ Pa. This will give a relative humidity $\phi(T) > 1$ which indicates that cloud form below this altitude. Therefore, we know that cloud forms between the altitudes of 1500 m and 2250m. Since $\phi(h = 1500) = 0.73$ and $\phi(h = 2250) = 1.35$, which indicates that the altitude that

Altitude (m)	p_a (atm)	Temperature (K)	p_{w} (Pa)	p_{sat} (Pa)	$\phi(T)$
0	1.00	293.15	819	2340	0.35
750	0.92	288.15	743	1700	0.44
1500	0.84	279.15	679	927	0.73

Table 1:

cloud formation occurs is somewhere in the middle of these two altitudes. A good guess would be around 2000 m.

(i) These regions should all have quite a lot of rain. At region B, the temperature above the sea water is cold so evaporation happens less frequently, but there are mountains in the Scotland region such that when wind blows the air uphill the air can quickly get to a high altitude and condense to form cloud and rain on the windward side. On the leeward side, however, the air becomes dry and there is less rain. At around location C, there are plenty of sea water to evaporate and the temperature is higher above the water surface so there are more water vapor in the air to begin with. When the air rises and cools through convection, cloud will also form and there will be rain. Region A may have less rain than C due to the air from continental Europe to be drier and the amount of water to evaporate is less, but there will still be rain.