

BPh O

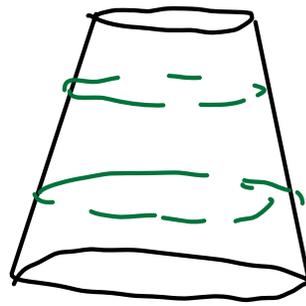
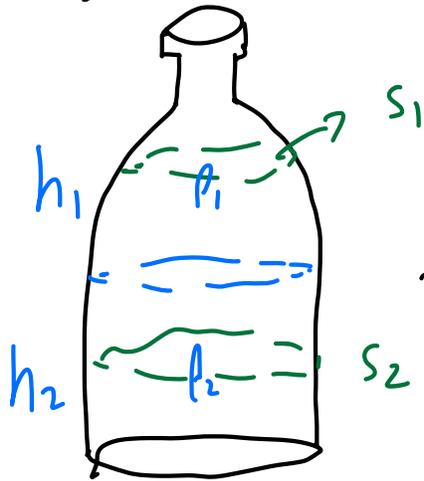
2021

Round 2

Solutions

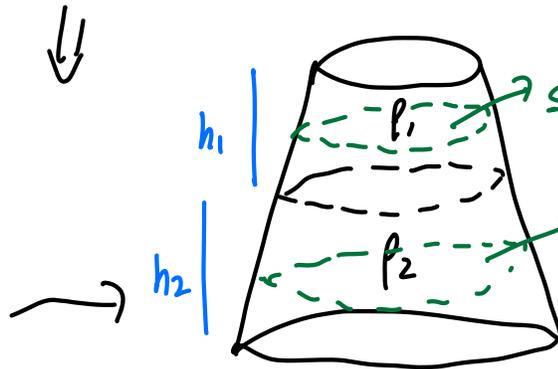
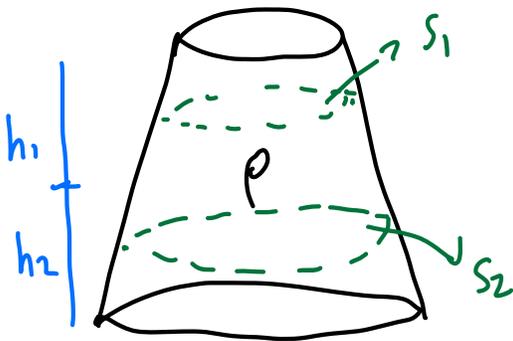
Ziyan Li

① (a)



↑ narrower on average

↓ wider on average



oil
vinegar
 $\rho_1 < \rho_2$

Average cross-sectional area.

$$S_1 < S_2$$

$$\rho_1 < \rho_2$$

$$p = \frac{\rho_1 S_1 h_1 + \rho_2 S_2 h_2}{S_1 h_1 + S_2 h_2}, \quad p = \rho g (h_1 + h_2); \quad p' = \rho_1 g h_1 + \rho_2 g h_2$$

$$\Rightarrow \frac{1}{g} [S_1 h_1 + S_2 h_2] (p - p') = (\rho_1 S_1 h_1 + \rho_2 S_2 h_2) (h_1 + h_2) - (\rho_1 h_1 + \rho_2 h_2) (S_1 h_1 + S_2 h_2)$$

$$= \cancel{\rho_1 S_1 h_1^2} + \rho_1 S_1 h_1 h_2 + \rho_2 S_2 h_2 h_1 + \cancel{\rho_2 S_2 h_2^2} - \cancel{\rho_1 S_1 h_1^2} - \rho_1 S_2 h_1 h_2 - \rho_2 S_1 h_2 h_1 - \cancel{\rho_2 S_2 h_2^2}$$

applies to more than 2 liquids

$$= h_1 h_2 (\rho_1 S_1 + \rho_2 S_2 - \rho_1 S_2 - \rho_2 S_1) = h_1 h_2 (\underbrace{\rho_2 - \rho_1}_{>0}) (\underbrace{S_2 - S_1}_{>0}) > 0$$

rearrangement inequality 排序不等式

$$\Rightarrow p - p' > 0 \Rightarrow p > p' \Rightarrow \text{pressure less than before}$$

OR use appropriate limits (初中物理, 极限法)

(d) Average K.E. : $K = \frac{3}{2} k_B T = \frac{1}{2} m v^2$

$\Rightarrow v = \sqrt{\frac{3 k_B T}{m}}$

for sodium $m = (1.66 \times 10^{-27}) \cdot (23)$

$\Rightarrow v = 1275 \text{ m/s}$

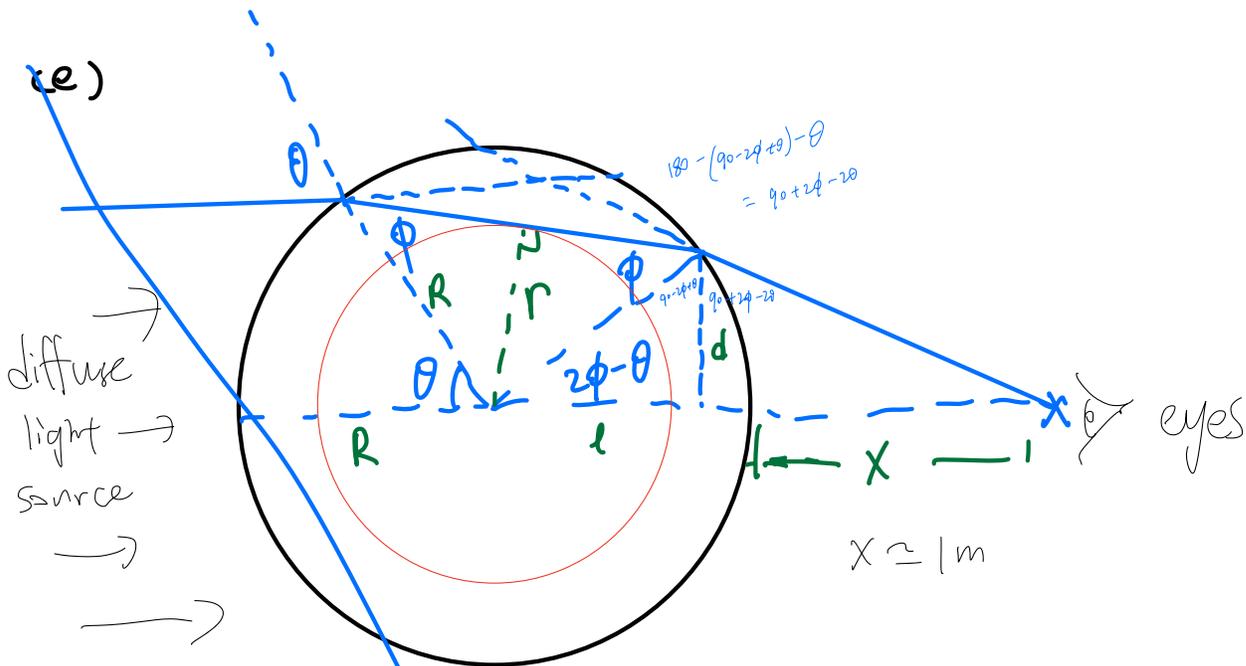
Doppler's effect : $|\delta f| = \frac{v}{c} f$

$\because \lambda f = c \Rightarrow \lambda df + f d\lambda = 0 \Rightarrow -\frac{df}{f} = \frac{d\lambda}{\lambda} = -\frac{v}{c}$

$\Rightarrow |\delta \lambda| = \frac{v}{c} \lambda$

$\Rightarrow \text{spread } \Delta \lambda = 2|\delta \lambda| = \frac{2v}{c} \lambda = \frac{2 \cdot 1275}{3 \times 10^8} \cdot 590 \text{ nm}$

$= 5.02 \times 10^{-3} \text{ nm}$



$R = 10 \text{ cm}$, for the flask to contain 1L of liquid,
inner radius r has to satisfy :

$$\frac{4}{3}\pi r^3 = 1000 \text{ cm}^3 \Rightarrow r = 6.2 \text{ cm}$$

so if $r \geq 6.2 \text{ cm}$, then the flask can.

The flask looks black all over \Rightarrow no light reflected from the inner surface (and thus the opaque liquid inside) can reach observer's eyes.

take $n=1.5$ for glass, assume max distance from flask to eyes $x=70 \text{ cm}$, for $r=6.2 \text{ cm}$, in the barely touching case we have

$$\sin \phi = \frac{6.2}{10} \Rightarrow \phi = 38.3^\circ$$

$$\sin \theta = n \sin \phi = 1.5 \sin(38.3) \Rightarrow \theta = 68.4^\circ$$

$$\Rightarrow l = R \cos(2\phi - \theta) = 9.9 \text{ cm}$$

$$d = R \sin(2\phi - \theta) = 1.43 \text{ cm}$$

$$x = d \tan(90^\circ + 2\phi - 2\theta) + l - R = 0.7 \text{ cm} \ll 70 \text{ cm} \therefore \text{OK}$$

$$\Rightarrow r \text{ can be } \geq 6.2 \text{ cm}$$

$$\Rightarrow \text{flask can hold 1L water}$$

BPhO 2021 Round 2 Question 2

Ziyan Li

Proton Decay - Solutions

- (a) Using the given condition that the rate of decay is proportional to the amount of original substance remains, we have

$$\frac{dN}{dt} = -\lambda N \quad (1)$$

which can be rearranged to

$$\int_{N_0}^{N(t)} \frac{dN'}{N'} = - \int_0^t dt' \quad (2)$$

which gives

$$N(t) = N_0 e^{-\lambda t} \quad (3)$$

as required.

(b)

- (i) The number of particles $|dN|$ that decays in time dt is given by

$$|dN| = \lambda N dt = \lambda N_0 e^{-\lambda t} dt \quad (4)$$

The probability dP for a proton to decay in this time interval $[t, t + dt]$ is the fraction of the total initial number of protons that decays during this specific time interval, which is

$$dP = \frac{|dN|}{N_0} = \lambda e^{-\lambda t} dt \quad (5)$$

as expected.

- (ii) The total probability is

$$\int dP = \int_0^\infty \frac{dP}{dt} dt = \int_0^\infty \lambda e^{-\lambda t} dt = 1 \quad (6)$$

- (iii) The probability for a particle to decay after time t is given by summing the disjoint probabilities

$$P(t) = \int_0^t \lambda e^{-\lambda t'} dt' = 1 - e^{-\lambda t} \quad (7)$$

So the probability for a particle to remain undecayed after time t is given by

$$p(t) = 1 - P(t) = e^{-\lambda t} \quad (8)$$

(iv) The lifetime is defined to be the time for a proton to remain undecayed. Hence it is given by

$$\tau = \frac{\int_0^{\infty} t e^{-\lambda t} dt}{\int_0^{\infty} e^{-\lambda t} dt} = \frac{(1/\lambda^2) \int_0^{\infty} x e^{-x} dx}{1/\lambda} = \frac{1/\lambda^2}{1/\lambda} = \frac{1}{\lambda} \quad (9)$$

where we've used the substitution $x = \lambda t$.

(c)

(i) The heat flux $Q = 92 \times 10^{-3} \text{ J}/(\text{s} \cdot \text{m}^2)$ is given by

$$Q = \frac{E}{A\tau} = \frac{N_0 E_p}{4\pi R_E^2 \tau} = \frac{M_E E_p}{4\pi R_E^2 m_p \tau} \quad (10)$$

where E is the total energy released from the Earth's surface, $N_0 = \frac{M_E}{m_p}$ is the number of protons the Earth contains, M_E and R_E are the mass and radius of the Earth, m_p and $E_p = 938 \text{ MeV}$ are the mass and rest energy of an individual proton, and $A = 4\pi R_E^2$ is the surface area of the Earth. Substituting in numbers gives

$$\tau = 1.14 \times 10^{28} \text{ s} = 3.6 \times 10^{20} \text{ years} \quad (11)$$

(ii) In reality, there will be other sources of heat outflux that constitute this total Q . So the part of Q that due to proton decay will be smaller than the value given in the question, which makes the calculated τ longer.

(d)

(i) Since only 40% of the decayed protons can be detected, we need $\frac{1}{0.4} = 2.5$ decayed protons on average to observe one decayed proton. On average if this takes time δt , then we have

$$(N_0 - 2.5) = N_0 e^{-\frac{\delta t}{\tau}} \quad (12)$$

which after rearrangement gives

$$N_0(1 - e^{-\frac{\delta t}{\tau}}) = 2.5 \quad (13)$$

For $\delta t \ll \tau$, we then have

$$N_0(1 - (1 - \frac{\delta t}{\tau})) = 2.5 \quad (14)$$

hence

$$\delta t = \frac{2.5\tau}{N_0} \quad (15)$$

Mass of a single water molecule is $(2 \times 8 + 2 \times 1) \times 1.66 \times 10^{-27} = 2.99 \times 10^{-26} \text{ kg}$. So 50000 tons of water contains 1.67×10^{33} water molecules. Since 1 water molecule contains $2 \times 1 + 8 = 10$ protons, the total number of protons in the water is $N_0 = 1.67 \times 10^{34}$. Substituting this in and use the τ we obtained from the previous estimate we have

$$\delta t = 1.7 \times 10^{-6} \text{ s} \quad (16)$$

(ii) On average there are

$$\mu = \frac{365 \times 24 \times 60 \times 60}{1.7 \times 10^{-6}} = 1.86 \times 10^{13} \quad (17)$$

decays to be observed per year.

(iii) Using the Poisson's distribution, the probability of not observing any decays in a year is

$$p(0) = \frac{\mu^0 e^{-\mu}}{0!} = e^{-\mu} \quad (18)$$

which is essentially 0. But we have not observed any proton decay so far. This means that the lifetime we calculated was a considerable underestimate. In fact, current beyond the standard model theories gives the lower bound of proton decay lifetime to be at least 10^{34} years.

(e) We set $c = 1$ and use the conservation of total energy, including rest energy and kinetic energy

$$E_p = E_e + E_\pi \quad (19)$$

and the conservation of momentum considering the proton is initial at rest

$$p_e = p_\pi \quad (20)$$

and the relativistic energy-momentum relation

$$E_p = m_p \quad (21)$$

and

$$E_e^2 = p_e^2 + m_e^2 \quad (22)$$

and

$$E_\pi^2 = p_\pi^2 + m_\pi^2 \quad (23)$$

Subtracting (23) by (22) gives

$$E_\pi^2 - E_e^2 = (E_\pi + E_e)(E_\pi - E_e) = m_p(E_\pi - E_e) = m_\pi^2 - m_e^2 \quad (24)$$

so we have

$$E_\pi - E_e = \frac{m_\pi^2 - m_e^2}{m_p} \quad (25)$$

Using addition and subtraction to solve (19) and (25) simultaneously and using the numbers of rest masses given in the question we have

$$E_\pi = \frac{m_p^2 + m_\pi^2 - m_e^2}{2m_p} = 478.7 \text{ MeV} \quad (26)$$

and

$$E_e = \frac{m_p^2 + m_e^2 - m_\pi^2}{2m_p} = 459.3 \text{ MeV} \quad (27)$$

Using $E^2 = p^2 + m^2$ and $p = \gamma mv$, it can be shown that $E = \gamma mc^2 = \gamma m$.

Since we know $E_\pi = \gamma_\pi m_\pi$ and $E_e = \gamma_e m_e$ we can solve for the γ 's to get $\gamma_\pi = 3.546$ and $\gamma_e = 898.8$. Using the relation

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (28)$$

we can solve for

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \quad (29)$$

and hence

$$v_\pi = 0.96c = 2.88 \times 10^8 \text{ m/s} \quad (30)$$

and

$$v_e = 0.99999938c \approx c \quad (31)$$

BPhO 2021 Round 2 Question 3

Ziyan Li

Orbits - Solutions

(a) Using the fact that the total energy of an elliptical orbit is conserved

$$E = \frac{1}{2}mv_a^2 - \frac{GMm}{r_a} = \frac{1}{2}mv_p^2 - \frac{GMm}{r_p} \quad (1)$$

and the conservation of angular momentum

$$mv_a r_a = mv_p r_p \quad (2)$$

we have

$$r_a^2 E = \frac{1}{2}m(v_a r_a)^2 - GMmr_a \quad (3)$$

and

$$r_p^2 E = \frac{1}{2}m(v_p r_p)^2 - GMmr_p \quad (4)$$

Subtracting the above two equations gives

$$(r_a + r_p)(r_a - r_p)E = -GMm(r_a - r_p) \quad (5)$$

For elliptical orbits, we have $r_a + r_p = 2a$ and $r_a - r_p \neq 0$, so we arrive at the expression for the total energy of an elliptical orbit

$$E = -\frac{GMm}{2a} \quad (6)$$

For an arbitrary point in the orbit we then have

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a} \quad (7)$$

which after rearrangement gives

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right) \quad (8)$$

as required.

(b)

(i) In order to maximise the new kinetic energy after the burn we need to align Δv with the direction of the original velocity. So we must have $v + \Delta v = v_e$ where $v_e = \sqrt{\frac{2GM}{r}}$ is the escape velocity. Substituting the expression for v in the previous question we have

$$\Delta v = \sqrt{GM} \cdot \left(\sqrt{\frac{2}{r}} - \sqrt{\left(\frac{2}{r} - \frac{1}{a} \right)} \right) \quad (9)$$

When $r = r_a = a(1 + e)$, we have

$$\Delta v_a = \sqrt{GM} \cdot \left(\sqrt{\frac{2}{a(1+e)}} - \sqrt{\frac{1-e}{a(1+e)}} \right) \quad (10)$$

and similarly

$$\Delta v_p = \sqrt{GM} \cdot \left(\sqrt{\frac{2}{a(1-e)}} - \sqrt{\frac{1+e}{a(1-e)}} \right) \quad (11)$$

(ii) Hence we have

$$\frac{\Delta v_a}{\Delta v_p} = \frac{\sqrt{\frac{2}{a(1+e)}} - \sqrt{\frac{1-e}{a(1+e)}}}{\sqrt{\frac{2}{a(1-e)}} - \sqrt{\frac{1+e}{a(1-e)}}} \quad (12)$$

Multiply both the numerator and the denominator by $\sqrt{a(1-e^2)}$ gives

$$\frac{\Delta v_a}{\Delta v_p} = \frac{\sqrt{2(1-e)} - (1-e)}{\sqrt{2(1+e)} - (1+e)} \quad (13)$$

as required

(iii) Consider the function

$$f(x) = \sqrt{\frac{2}{x}} - \sqrt{\frac{2}{x} - \frac{1}{a}} \quad (14)$$

in the range $x \in (0, 2a)$ for $a > 0$. We have

$$\frac{df}{dx} = \frac{1}{x^2} \left(\frac{1}{\sqrt{\frac{2}{x} - \frac{1}{a}}} - \frac{1}{\sqrt{\frac{2}{x}}} \right) \quad (15)$$

Since $0 < x < 2a$ we have $\sqrt{\frac{2}{x} - \frac{1}{a}} < \sqrt{\frac{2}{x}}$ always and hence $\frac{df}{dx} > 0$ in the full range of function $f(x)$. We can conclude that $f(x)$ is monotonically increasing in its full range.

The expression for Δv in (i) can thus be re-written as

$$\Delta v(r) = \sqrt{GM} \cdot f(r) \quad (16)$$

where $r \in [a(1-e), a(1+e)]$. Since $0 < e < 1$ this is indeed a sub-range of $(0, 2a)$. So $f(r)$ is of course monotonically increasing in the range. Hence $r_a > r_p$ implies we have $\Delta v_a = \Delta v(r_a) > \Delta v(r_p) = \Delta v_p$, which means we need a greater speed increase from the apogee than from the perigee.

(c) The original speed at perigee (first burn point) is

$$v_p = \sqrt{\frac{GM}{r_1}} \quad (17)$$

The first burn makes the orbit an ellipse of semi-major axis such that $2a = r_1 + r_2$. Using the Newton's *vis viva* equation we derived in (a) we get

$$v_1 = v_p + \Delta v_p = \sqrt{GM \left(\frac{2}{r_1} - \frac{2}{r_1 + r_2} \right)} = \sqrt{\frac{GM}{r_1}} \left(\sqrt{\frac{2r_2}{r_1 + r_2}} \right) \quad (18)$$

hence

$$\Delta v_p = v_1 - v_p = \sqrt{\frac{GM}{r_1}} \left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right) \quad (19)$$

as required. After the second burn, the orbit becomes a circle of radius r_2 with speed

$$v_a = \sqrt{\frac{GM}{r_2}} \quad (20)$$

At the position of the apogee (the second burn), the speed v_2 is given by the conservation of angular momentum $mv_1r_1 = mv_2r_2$ so we have

$$v_2 = \frac{r_1}{r_2}v_1 = \sqrt{\frac{GM}{r_2}} \left(\sqrt{\frac{2r_1}{r_1 + r_2}} \right) \quad (21)$$

Therefore the change in speed at the apogee is

$$\Delta v_a = v_a - v_2 = \sqrt{\frac{GM}{r_2}} - \sqrt{\frac{GM}{r_2}} \left(\sqrt{\frac{2r_1}{r_1 + r_2}} \right) \quad (22)$$

which can be simplified to

$$\Delta v_a = \sqrt{\frac{GM}{r_2}} \left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}} \right) \quad (23)$$

as required.

(d) The total change in speed is

$$\Delta v = \Delta v_a + \Delta v_p = \sqrt{GM} \left(\sqrt{\frac{2r_2}{r_1(r_1 + r_2)}} - \sqrt{\frac{1}{r_1}} + \sqrt{\frac{1}{r_2}} - \sqrt{\frac{2r_1}{r_2(r_1 + r_2)}} \right) \quad (24)$$

where r_2 is the only variable and we differentiate to get

$$\frac{d\Delta v}{dr_2} = \sqrt{\frac{GM}{r_1}} \left[-\frac{1}{2} \left(\frac{1}{x} \right)^{\frac{3}{2}} \left(1 - \sqrt{\frac{2}{1+x}} \right) + \sqrt{\frac{1}{2x}} \left(\frac{1}{1+x} \right)^{\frac{3}{2}} - \sqrt{\frac{x}{2}} \left(\frac{1}{1+x} \right)^{\frac{3}{2}} + \sqrt{\frac{1}{2x}} \sqrt{\frac{1}{1+x}} \right] \quad (25)$$

where $x = \frac{r_2}{r_1}$. We need $\frac{d\Delta v}{dr_2} = 0$, so we require the RHS to be 0. Multiplying the RHS by $\sqrt{\frac{2r_1}{GM}}(x(1+x))^{3/2}$ gives

$$-\frac{\sqrt{2}}{2}(1+x)^{3/2} + (1+x) + x - x^2 + x(1+x) = 0 \quad (26)$$

which simplifies to

$$(1+x)^{3/2} = \sqrt{2}(1+3x) \quad (27)$$

Squaring both sides gives

$$(x+1)^3 = 2(1+3x)^2 \quad (28)$$

which yields

$$x^3 - 15x^2 - 9x - 1 = 0 \quad (29)$$

as required.

BPhO 2021 Round 2 Question 4

Ziyan Li

EM Waves - Solutions

(a) From the equation

$$E_z = E_{z0} \sin(kx - \omega t) \quad (1)$$

When looking at wave propagation we need the same field strength to be present at coordinates (t, x) and $(t + \Delta t, x + \Delta x)$ so

$$\sin(kx - \omega t) = \sin(k(x + \Delta x) - \omega(t + \Delta t)) \quad (2)$$

which, using angle sum formulae can be simplified to

$$\tan(kx - \omega t)(1 - \cos(k\Delta x - \omega\Delta t)) = \sin(k\Delta x - \omega\Delta t) \quad (3)$$

By observation the solution is clearly

$$c = \frac{\Delta x}{\Delta t} = \frac{\omega}{k} \quad (4)$$

is the speed of propagation of the wave and since this number is positive the direction of propagation is clearly in the positive direction.

(b) The identical wave traveling to the negative direction is

$$E_{z0} \sin(kx + \omega t) \quad (5)$$

and their superposition is

$$\begin{aligned} E'_z &= E_{z0}(\sin(kx - \omega t) + \sin(kx + \omega t)) \\ &= E_{z0}(\sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t) + \sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega t)) \\ &= 2E_{z0}\sin(kx)\cos(\omega t) \end{aligned} \quad (6)$$

which is a standing wave as required, where $E'_{z0} = 2E_{z0}$.

(c)

(i) For the nodes we require $\sin(kx) = 0$ which gives $kx = n\pi$. So the distance between adjacent nodes is

$$\delta x = \frac{\pi}{k} = \frac{\lambda}{2} = \frac{c}{2f} = \frac{3 \times 10^8}{2 \times 60 \times 10^6} = 2.5 \text{ m} \quad (7)$$

where we have used $k = \frac{2\pi}{\lambda}$ and $c = \frac{\omega}{k} = \lambda f$.

(ii) The intensity of the waves decreases with distance from the source due to spreading. This means that the amplitude of the standing wave will vary across the space, which can lead to a less uniform pattern.

(d)

(i) Monochromatic light ensures the same distinct frequency which can guarantees interference. The photosensitive film was much thinner than one wavelength can make sure that the change in index of refraction within the film has little effect on the overall optical path. The photographic film material was not reflective can make sure that only the reflected wave from the mirror interferes with the original wave to produce standing wave, and there is no other reflected waves that interferes.

(ii) The wave travelling through the apparatus is shown in the figure. It is known that reflected

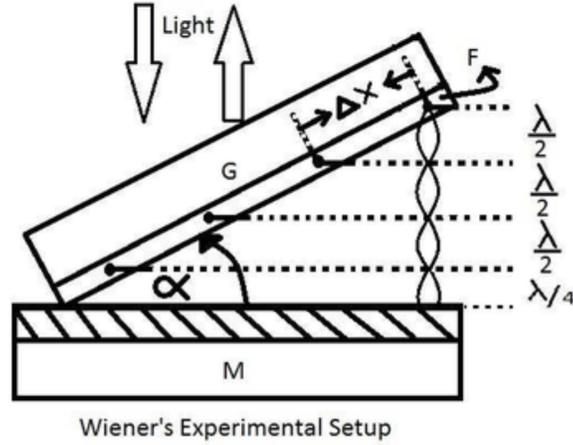


Figure 1: Otto Wiener's experiment

light will experience a 180 degree phase change when it reflects from a medium of higher index of refraction and no phase change when it reflects from a medium of smaller index. This is why just above the mirror a total path of $\frac{\lambda}{2}$ (one $\frac{\lambda}{4}$ going into the mirror direction and one $\frac{\lambda}{4}$ going out of the mirror) is shifted out and the physical lengths in the apparatus are as shown in the figure. From geometry it can be deduced that for small angle $\sin \alpha \approx \alpha$ we have

$$\delta x = \frac{\lambda}{2\alpha} = \frac{400 \times 10^{-9}}{2 \cdot \frac{1}{30}} = 6 \times 10^{-6} \text{ m} \quad (8)$$

(e)

(i),(ii) Note that

$$\begin{aligned} & \cos(a + b) + \cos(a - b) \\ &= \cos a \cos b - \sin a \sin b + \cos a \cos b + \sin a \sin b \\ &= 2 \cos a \cos b \end{aligned} \quad (9)$$

so we have

$$\begin{aligned} E''_z &= E_{z0}(\cos(\omega_1 t) + \cos(\omega_2 t)) \\ &= 2E_{z0} \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \end{aligned} \quad (10)$$

as required, where $E''_{z0} = 2E_{z0}$, $\omega_a = \frac{\omega_1 + \omega_2}{2}$ and $\omega_b = \frac{\omega_1 - \omega_2}{2}$

- (iii) If the two frequencies are close to each other, the shape of this beats pattern is a fast oscillation of the modulus enclosed by a slow oscillation of an envelope. Because every other burst in the modulation pattern is inverted, each peak is replaced by a trough and vice versa. The envelope is perceived to have twice the frequency of the modulating cosine, which means the beats frequency is

$$f = 2 \times \frac{1}{2\pi} |\omega_b| = |f_1 - f_2| = 10 \text{ Hz} \quad (11)$$

- (iv) The display is $0.2 \times 10 = 2 \mu\text{s}$ in time range. The period of the beats envelope is given by $T = \frac{1}{1\text{MHz}} = 1 \mu\text{s}$. This means that the period of the envelope cosine function displayed on the

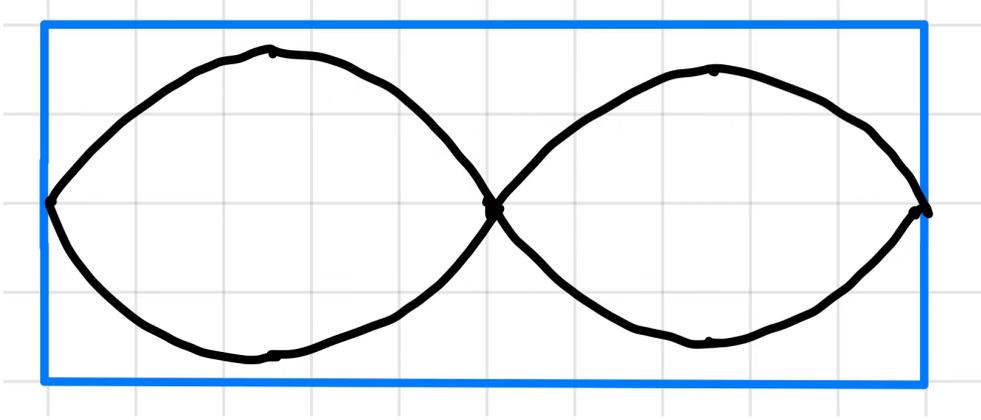


Figure 2: The display of the oscilloscope

oscilloscope is $\frac{1}{0.2} = 5$ squares in time. The display of 10 squares of time on the oscilloscope is shown in the figure.

- (v) Since $f = \frac{1}{T}$, we have $\ln f = -\ln T$. Differentiate this we have

$$\frac{df}{f} = -\frac{dT}{T} \quad (12)$$

So percentage error in period measurement is roughly the same as the percentage error in frequency measurement. This gives a percent error of around 5%.

- (vi) In the green regime a typical wavelength is about $\lambda = 550 \text{ nm}$, so a typical frequency is around

$$f = \frac{3 \times 10^8}{550 \times 10^{-9}} = 5.45 \times 10^{14} \text{ Hz} = 5.45 \times 10^8 \text{ MHz} \quad (13)$$

Hence to measure a difference of 1 MHz, we need 9 significant figures.

(f)

- (i) Following small angle approximation, the equivalent of angle α is $\frac{a}{D}$. Under the assumption that $a \ll D$ and $x \ll D$, standard Young's Double Slit derivation gives

$$a \sin \theta = a\theta = a \frac{my}{D} = m\lambda \quad (14)$$

which gives

$$y = \frac{\lambda D}{a} \quad (15)$$

- (ii) The analogy does not hold perfectly and in Wiener's experiment, when the path difference is zero there is a dark point. This is because the 180 degree phase shift that arises when wave is reflected as it travels from a medium with lower index of refraction to a medium with higher index of refraction.
- (iii) Consider the screen to be the y - z plane and the line joining the two slits to be parallel to the z -axis. Also we assume that the two slits have their lengths negligible compare to a , i.e. the two slits can be treated as two point pinhole sources.

We assign the coordinates of the two slits to be $(-D, 0, -\frac{a}{2})$ and $(-D, 0, \frac{a}{2})$, and an arbitrary point on one of the bright fringes to be $(0, y, z)$, then the path difference for constructive interference is given by

$$\left| \sqrt{D^2 + y^2 + (z + \frac{a}{2})^2} - \sqrt{D^2 + y^2 + (z - \frac{a}{2})^2} \right| = m\lambda \quad (16)$$

Rearranging and squaring both sides gives

$$D^2 + y^2 + (z + \frac{a}{2})^2 = D^2 + y^2 + (z - \frac{a}{2})^2 \pm 2m\lambda \sqrt{D^2 + y^2 + (z - \frac{a}{2})^2} + (m\lambda)^2 \quad (17)$$

Further rearranging gives

$$2za - (m\lambda)^2 = \pm 2m\lambda \sqrt{D^2 + y^2 + (z - \frac{a}{2})^2} \quad (18)$$

Again squaring both sides gives

$$4z^2a^2 - 4za(m\lambda)^2 + (m\lambda)^4 = 4D^2(m\lambda)^2 + 4y^2(m\lambda)^2 + 4z^2(m\lambda)^2 - 4za(m\lambda)^2 + a^2(m\lambda)^2 \quad (19)$$

And further rearrangement gives

$$4z^2(a^2 - (m\lambda)^2) - 4y^2(m\lambda)^2 = 4D^2(m\lambda)^2 + (m\lambda)^2(a^2 - (m\lambda)^2) \quad (20)$$

Divide out common factors yields

$$\left(\frac{a^2}{m^2\lambda^2} - 1\right)z^2 - y^2 = D^2 + \frac{a^2 - m^2\lambda^2}{4} \quad (21)$$

which is indeed a hyperbola.

We need $a \ll D$ because if $a \sim D$ then the difference in lengths between two paths is comparable with the lengths of these two paths, which means we can no longer treat the two superposing waves as having the same intensity. So in this case the interference may not be perfectly constructive when the path difference is a constant integer multiple of wavelength.