BPhO

2020

Round 2

Solutions

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(1)

(a) $V = l^3$, $V_0 = l_0^3$

$\Delta V = V - V_0 = l^3 - l_0^3 = \left[(1+\alpha\Delta T)^3 - 1\right] l_0^3$

$\simeq (1 + 3\alpha\Delta T - 1) V_0 = 3\alpha\Delta T\, V_0$

$\Rightarrow V = (1 + 3\alpha\Delta T) V_0 \Rightarrow \underline{\gamma = 3\alpha}$

(b) (i)

i $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$, $r = r_1 + r_2$

ii $\mathcal{E} = \dfrac{\mathcal{E}_1/r_1 + \mathcal{E}_2/r_2}{1/r_1 + 1/r_2}$, $r = \dfrac{r_1 r_2}{r_1 + r_2}$

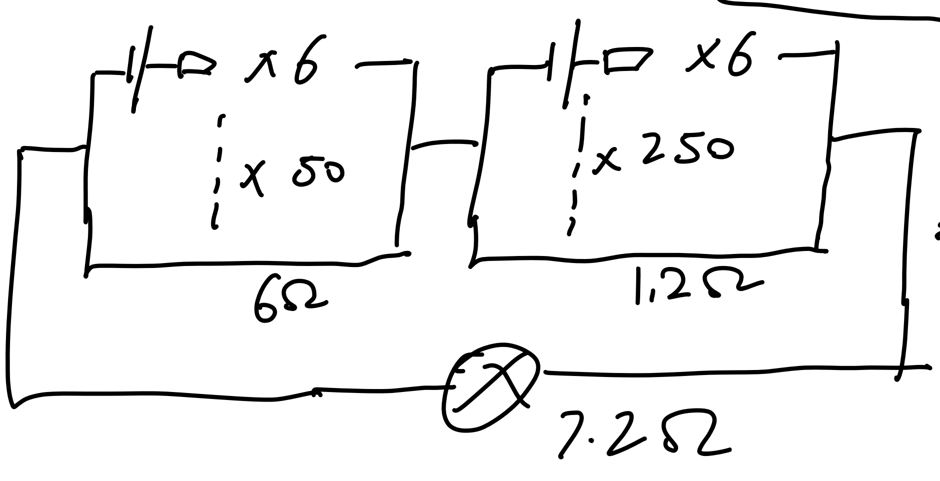
$r_1 = r_2 = r_0$, $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_0$

$\Rightarrow \mathcal{E} = \frac{1}{2}(\mathcal{E}_0 + \mathcal{E}_0) = \mathcal{E}_0$, $r = \frac{r_0}{2}$

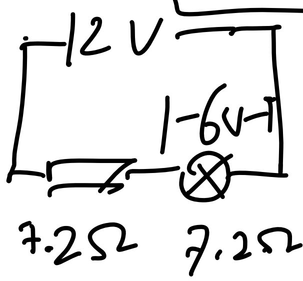
(ii) $P = \dfrac{V^2}{R} \Rightarrow R = \dfrac{V^2}{P} = 7.2\,\Omega$

| — 12V — | $V = 6V$

$\boxed{300 \times 6 = 1800}$



$\Rightarrow$



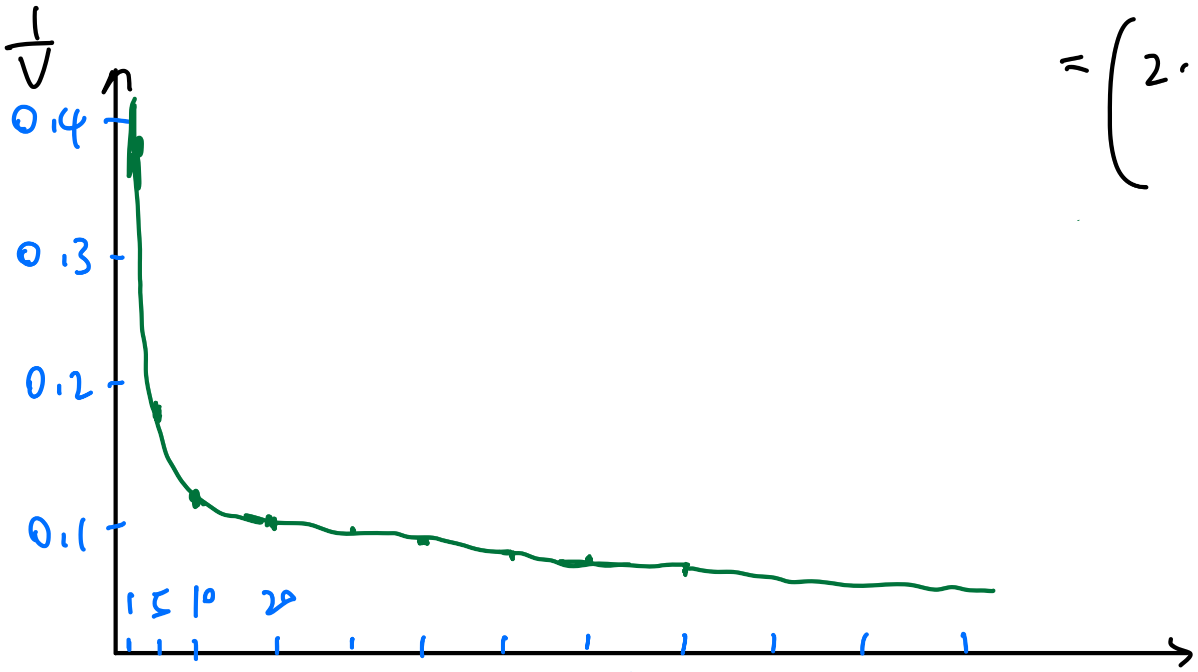
(c) - Crystal ice forms through diffusion

- E-fields polarise the water molecules
- field & field gradient are strong at sharp points
- alignment enhanced and tips sharper

(d) $a(x) = v\frac{dv}{dx} \Rightarrow \int v\,dv = \int a(x)\,dx$

$\Rightarrow \frac{1}{2}v^2 - 0 = \int a(x)\,dx \Rightarrow v = \left(2\int_0^x a\,dx'\right)^{\frac{1}{2}}$

$= \left(2 \cdot \text{Area under } a\text{-}x \text{ curve}\right)^{\frac{1}{2}}$



x:	0	10	20	30	40	50	60	70	80	90	100	5	1
a:	3.5	2.2	1.4	0.8	0.55	0.45	0.35	0.3	0.25	0.25	0.25	2.8	3.4
v:	0	7.55	9.64	10.72	11.34	11.76	12.10	12.36	12.59	12.79	12.98	5.61	2.62
$\frac{1}{v}$:	∞	0.132	0.104	0.093	0.088	0.085	0.083	0.081	0.079	0.078	0.077	0.178	0.382
t:	0	2.651	3.831	4.816	5.74	X	X	X	X	X	X	1.876	0.756

$$v(x) = \frac{dx}{dt} \Rightarrow \int_0^t dt' = \int_0^x \frac{dx'}{v(x')} \Rightarrow t = \int_0^x \frac{dx'}{v(x')}$$

assume $x: 0 \rightarrow 1$ $\quad a = 3.5\,m/s^2$

$$\frac{1}{2}(3.5)t^2 = 1 \Rightarrow t = 0.756s$$

from $X = 1 \rightarrow 100$, use area under $\frac{1}{v} - x$ graph

From the chart, we see $t = 5s$ is attained

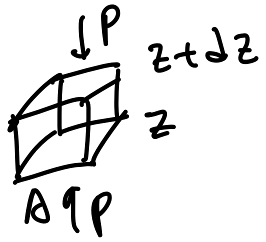
when $v \in (10.72, 11.34)$

$a \in (0.8, 0.55)$

$\Rightarrow$ estimate as $t = 5s$: $\boxed{\begin{aligned} v &= 10.8\,m/s \\ a &= 0.75\,m/s^2 \end{aligned}}$

$$\Rightarrow \frac{P}{m} = \frac{Fv}{m} = av = \underline{8.1\,W/kg}$$

(2) (a)



$$P(z)A = \rho A\,dz\,g + P(z+dz)A$$

$$\Rightarrow \frac{P(z+dz) - P(z)}{dz} = -\rho g$$

$$\Rightarrow \frac{dP}{dz} = -\rho g$$

(b) (i) $PV = Nk_BT \rightarrow P = nk_BT \rightarrow P = \frac{\rho}{m}k_BT$

$$\Rightarrow \rho = \frac{mP}{k_BT} \Rightarrow \frac{dP}{dz} = -\frac{mg}{k_BT}P \Rightarrow \lambda = \frac{mg}{k_BT}$$

$$\therefore \frac{dP}{dz} = -\lambda P \Rightarrow \int_{P_0}^{P(z)} \frac{dP}{P} = \int_0^z -\lambda\,dz$$

$$\Rightarrow \ln\left(\frac{P(z)}{P_0}\right) = -\lambda z \Rightarrow \underline{P(z) = P_0 e^{-\frac{mg}{k_BT}z}}$$

(ii)
$$g = \frac{GM_E}{(R_E+z)^2} = \frac{GM_E}{R_E^2}\left(1+\frac{z}{R_E}\right)^{-2}$$

$$\simeq \frac{GM_E}{R_E^2}\left(1 - \frac{2z}{R_E} + O\left(\left(\frac{z}{R_E}\right)^2\right)\right)$$

for $z \ll R_E$, $g \sim$ constant.

(iii) T is not constant as z increases.

(C) (i) $PV^\gamma = \text{const}$ $\quad PV = Nk_BT$

$\Rightarrow (PV)V^{\gamma-1} = \text{const} \Rightarrow Nk_BTV^{\gamma-1} = \text{const}$

$\Rightarrow TV^{\gamma-1} = \text{const}.$ (N stays the same)

$\Rightarrow p^{\gamma-1}V^{\gamma(\gamma-1)} = \text{const}$ $\Rightarrow p^{1-\gamma}T^\gamma = \text{const}.$

$T^\gamma V^{\gamma(\gamma-1)} = \text{const}$ $\quad = p_0^{1-\gamma}T_0^\gamma$

Now use $p = \frac{\rho}{m}k_BT$, $\quad \frac{dp}{dz} = -\rho g$

$\rho = \frac{mp}{k_BT} \Rightarrow \frac{dp}{dz} = -\frac{mgp}{k_BT}$

$\Rightarrow T\frac{dp}{p} = -\frac{mg}{k_B}dz$ $\boxed{1}$

$\because p^{1-\gamma}T^\gamma = 0 \Rightarrow (1-\gamma)\ln p + \gamma\ln T = 0$

$\Rightarrow (1-\gamma)\frac{dp}{p} + \gamma\frac{dT}{T} = 0 \Rightarrow \frac{dp}{p} = \frac{\gamma}{\gamma-1}\frac{dT}{T}$

$\Rightarrow T\frac{dp}{p} = \frac{\gamma}{\gamma-1}dT$ $\boxed{2}$

$\boxed{2} \to \boxed{1} \Rightarrow \frac{dT}{dz} = -\frac{\gamma-1}{\gamma}\frac{mg}{k_B} = -\frac{C_p-C_v}{C_p}\frac{mg}{k_B} = -\frac{Rmg}{C_pk_B}$

(ii)

$\Rightarrow T = T_0 - \frac{Rmg}{C_pk_B}z$ $\quad$ let $\frac{mg}{k_B} = \beta$

$$\Rightarrow \frac{dp}{dz} = -\beta\frac{p}{T} = -\beta\frac{p}{T_0 - \frac{R}{c_p}\beta z}$$

$$\Rightarrow \int_{p_0}^{p(z)} \frac{dp}{p} = -\frac{c_p}{R}\int_0^z \frac{dz'}{\frac{c_p T_0}{R\beta} - z'}$$

$$\Rightarrow \ln\left(\frac{p(z)}{p_0}\right) = \frac{c_p}{R}\ln\left(\frac{\frac{c_p T_0}{R\beta} - z}{\frac{c_p T_0}{R\beta}}\right)$$

$$\frac{p(z)}{p_0} = \left(\frac{\frac{c_p T_0}{R\beta} - z}{\frac{c_p T_0}{R\beta}}\right)^{\frac{c_p}{R}} \quad , \quad \beta = \frac{mg}{k_B}$$

$$p(z) = p_0\left(1 - \frac{R}{c_p}\left(\frac{mg}{k_B T_0}\right)z\right)^{\frac{c_p}{R}} \qquad \left(\frac{R}{c_p} = \frac{\gamma - 1}{\gamma}\right)$$

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* if $\gamma \to 1$, $\frac{R}{c_p} \to 0$. $\frac{c_p}{R} \to \infty$, let $\frac{c_p}{R} = x$

$$p(z) \to \lim_{x\to\infty} p_0\left(1 - \frac{1}{x}\left(\frac{mg}{k_B T_0}\right)z\right)^x = p_0 e^{-\frac{mg}{k_B T_0}z}$$ recoves the result in (i)

$$\left(\lim_{x\to\infty}\left(1 - \frac{a}{x}\right)^x = e^{-a}\right)$$

(d) μ is the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$$\omega = \sqrt{\frac{k}{\mu}} \Rightarrow \omega = 2\pi f \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

(e) wave number $k = \frac{2\pi}{\lambda}$, $k = 2\pi \nu$

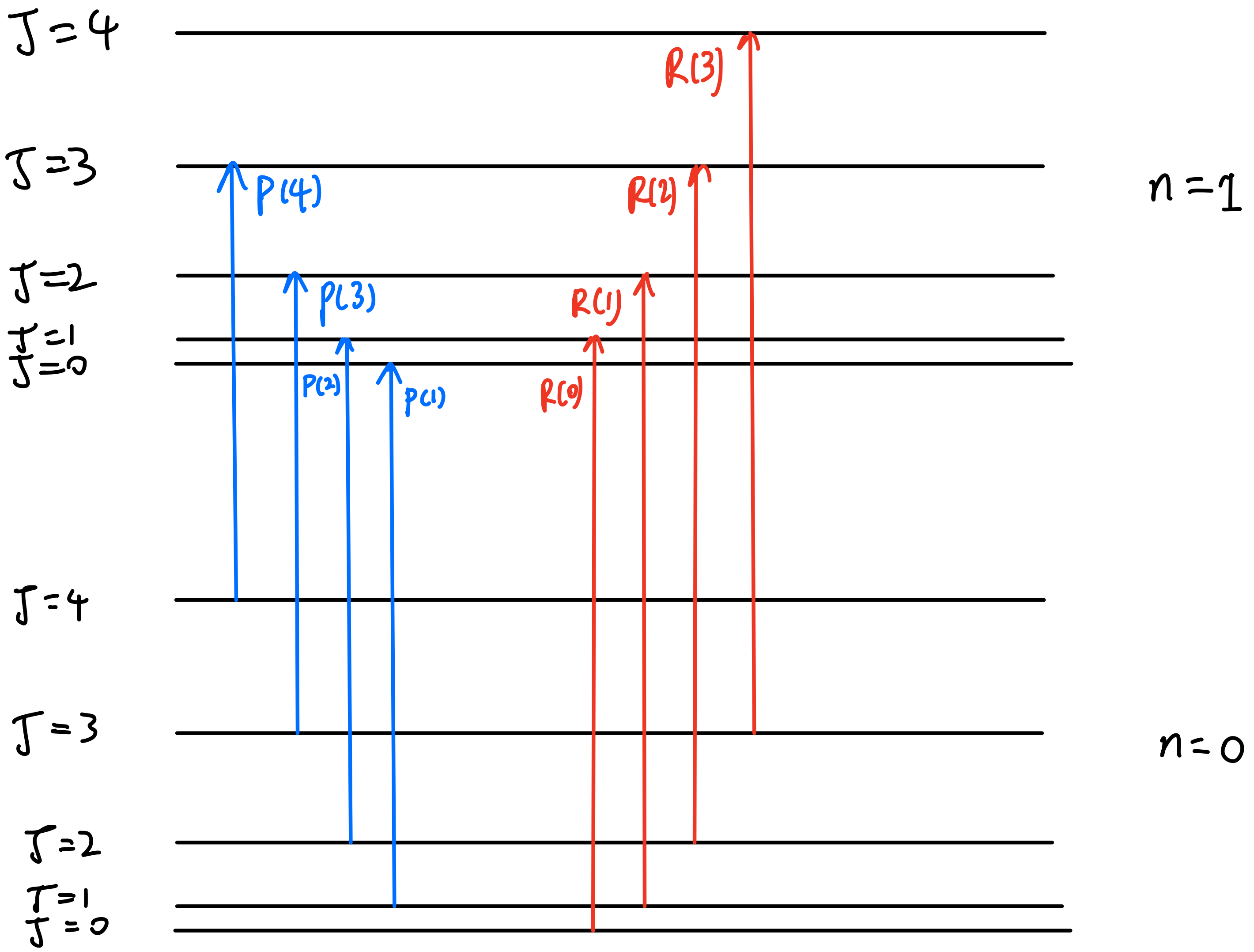
$$\Rightarrow \nu = \frac{1}{\lambda} \quad \text{(convention)}$$

$$E = hf = \frac{hc}{\lambda} = hc\nu = (6.63 \times 10^{-34})(3.0 \times 10^{8})(2900 \times \frac{1}{0.01})$$

$$= \underline{5.77 \times 10^{-20} J}$$

$$E = k_B T \Rightarrow T = \frac{E}{k_B} = \underline{4.18 \times 10^{3} K}$$

(f)



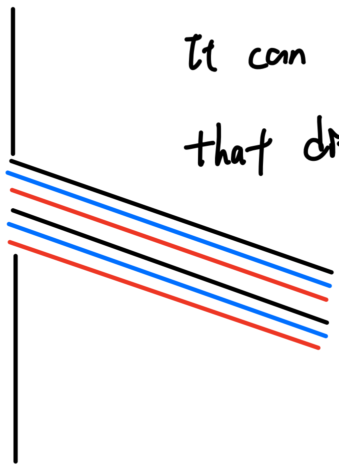
(g) $f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}} \Rightarrow k = 4\pi^2 f^2 \mu$

$$\mu = \frac{m_H m_{Cl}}{m_H + m_{Cl}} = 0.97957\,u = 1.6271 \times 10^{-27}\,kg$$

$$k = (4\pi^2)(8.66 \times 10^{13})^2 (1.6271 \times 10^{-27}) = 481.7\ N/m$$

③ (a) (i) secondary sources on old wavefronts creat new wavefronts, and the direction of propagation is $\perp$ to all wave fronts.

(ii)

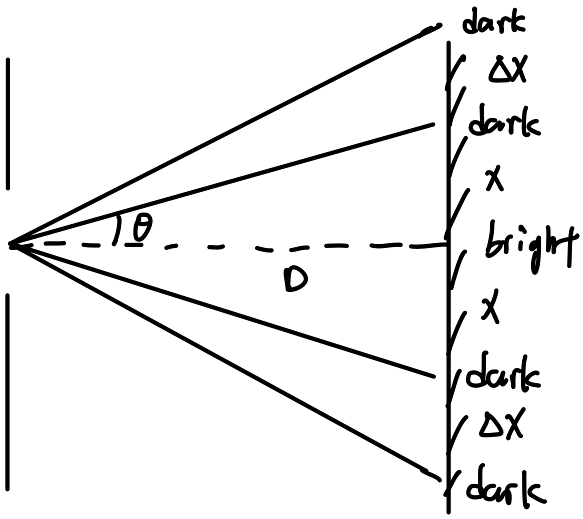


It can be seen that every pair of light wave that differs by path difference $\frac{\lambda}{2}$ cancels, so net intensity $= 0$.

(iii) That is shown by pairs of rays of different colors.

$\frac{1}{2} S_0 \sin\theta = m \cdot \frac{\lambda}{2} \Rightarrow S_0 \sin\theta = m\lambda \quad (m = \pm 1, \pm 2, \ldots \; m \neq 0)$

(iv) For $D \gg S_0$, $\sin\theta \simeq \tan\theta = \frac{x}{D}$



$\Rightarrow S_0 \frac{x}{D} = \lambda \Rightarrow x = \frac{\lambda D}{S_0}$

$\Rightarrow$ width $= 2x = \frac{2\lambda D}{S_0}$

$S_0 \frac{(x + \Delta x)}{D} = 2\lambda \Rightarrow$

spacing $= \Delta x = \frac{\lambda D}{S_0}$

(b) (i) Consider the wave $A\cos(\omega t + \phi(\delta s))$, If the screen is far enough from the aperture, $\phi(\delta s)$ can be approximated to a linear function with $\phi(0) = 0$ at the centre. This is Fraunhofer diffraction.

Thus we have $\phi(\delta s) = -\phi(-\delta s)$ and for every pair of $\pm\delta s$,

$$A\cos(\omega t + \phi(\delta s)) + A\cos(\omega t + \phi(-\delta s))$$

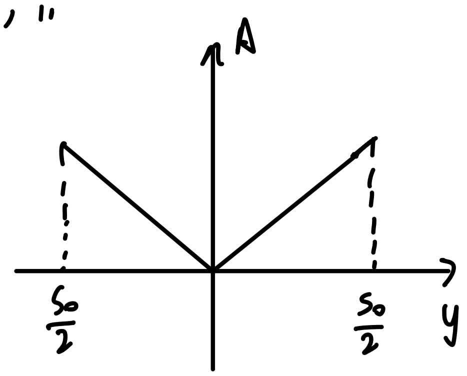
$$= A\cos\omega t\cos(\phi(\delta s)) - \cancel{A\sin\omega t\sin(\phi(\delta s))} + A\cos\omega t\cos(-\phi(\delta s))$$

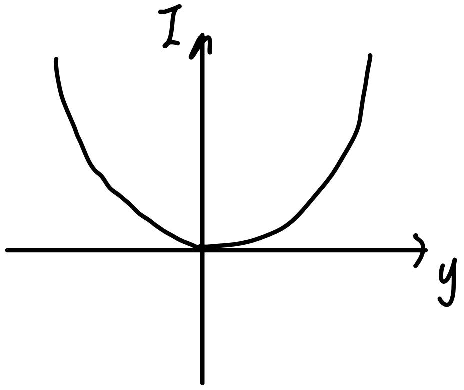
$$+ \cancel{A\sin\omega t\sin(\phi(\delta s))} = 2A\cos(\phi(\delta s))\underbrace{[\cos\omega t]}_{=\cos(\omega t+\delta(0)) \text{ at centre}}$$

$\Rightarrow$ in phase with centre (if $\phi(\delta s)$ linear)

(ii) At slit :

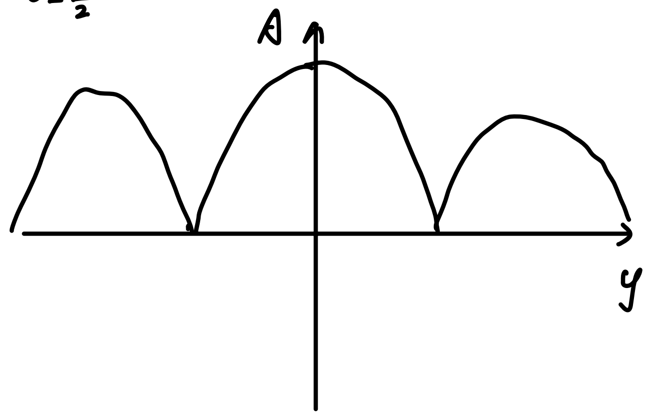
i, ii

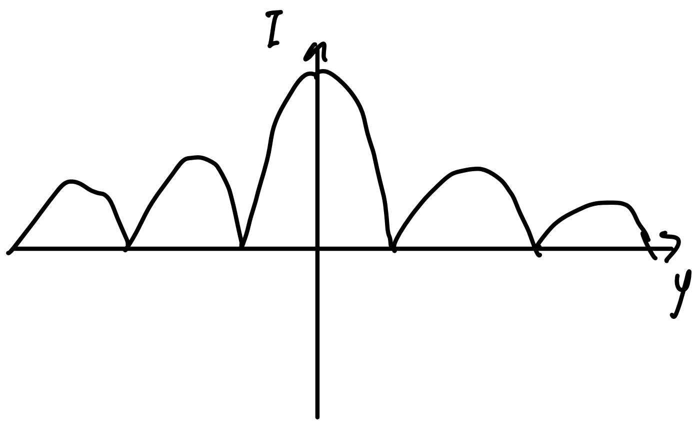




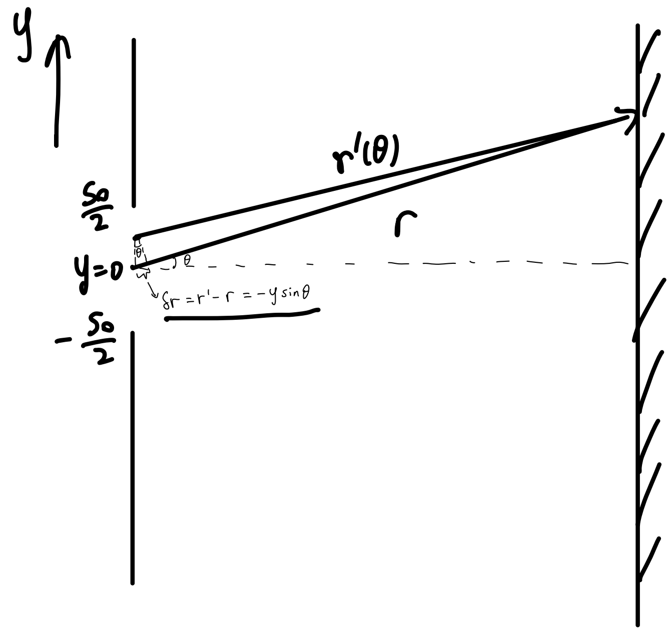
At screen :

$$\left(\int_{-\frac{a}{2}}^{\frac{a}{2}} |x|\, e^{-ibx}\, dx = \frac{ab\sin(\frac{ab}{2}) + 2\cos(\frac{ab}{2}) - 2}{b^2}\right)$$





(iii) Amplitude at aperture : A_0 per unit dy



$k = \frac{2\pi}{\lambda}$, wave $\sim e^{i(kr'(\theta) - \omega t)}$

Amplitude $dA = A_0 dy\, e^{i(kr'(\theta)-\omega t)}$

$= A_0 dy\, e^{ikr} e^{i(k\underbrace{\delta r}_{-y\sin\theta} - \omega t)}$

$$\Rightarrow \quad A(\theta) = A_0 e^{i(kr-\omega t)} \int_{-\frac{S_0}{2}}^{\frac{S_0}{2}} dy\, e^{-iky\sin\theta}$$

$$= A_0 e^{i(kr-\omega t)} \frac{1}{+\cancel{i}k\sin\theta}\left[\underbrace{e^{-ik\sin\theta\left(\frac{S_0}{2}\right)} - e^{-ik\sin\theta\left(-\frac{S_0}{2}\right)}}_{=\,+2\cancel{i}\sin\left(\frac{1}{2}kS_0\sin\theta\right)}\right]$$

in phase with wave from centre

$$= \underbrace{A_0 S_0}_{\text{total Amplitude at apercature}} e^{i(kr-\omega t)} \frac{\sin\left(\frac{1}{2}kS_0\sin\theta\right)}{\frac{1}{2}kS_0\sin\theta}$$

Define $I(0) = |A_0 S_0|^2$, use $k = \frac{2\pi}{\lambda} \Rightarrow \frac{1}{2}k = \frac{\pi}{\lambda}$ we have

$$I(\theta) = |A(\theta)|^2 = I(0)\frac{\sin^2\left(\frac{\pi S_0}{\lambda}\sin\theta\right)}{\left(\frac{\pi S_0}{\lambda}\sin\theta\right)^2}$$ as required.

(C) (i) We look for $\frac{d}{dx}\left(\frac{\sin x}{x}\right) = 0$

$$\Rightarrow \frac{x\cos x - \sin x}{x^2} \Rightarrow \quad x\cos x - \sin x = 0 \Rightarrow x = \tan x$$

first non-zero positive solution $x = 4.4934$

$$\Rightarrow \quad x = \frac{\pi S_0}{\lambda}\sin\theta = 4.4934$$

$$\Rightarrow \quad \sin\theta = 4.4934 \cdot \frac{\lambda}{\pi S_0} = 0.045$$

$$\sin\theta \simeq \tan\theta \simeq \theta = \frac{\Delta y}{D} \Rightarrow \Delta y = D\theta = 0.045 \cdot 1\,m = \underline{0.045\,m}$$

(ii)

$$\lim_{x \to 0}\left(\frac{\sin x}{x}\right)^2 = 1 \,, \quad \left(\frac{\sin(4.4934)}{4.4934}\right)^2 = 0.047$$

$\rightarrow$ first maximum $\underline{4.7\%}$

$\rightarrow$ second maximum $x = \frac{2\pi + 3\pi}{2} = \frac{5}{2}\pi$

$$\rightarrow \left(\frac{\sin x}{x}\right)^2 = 1.6\%$$

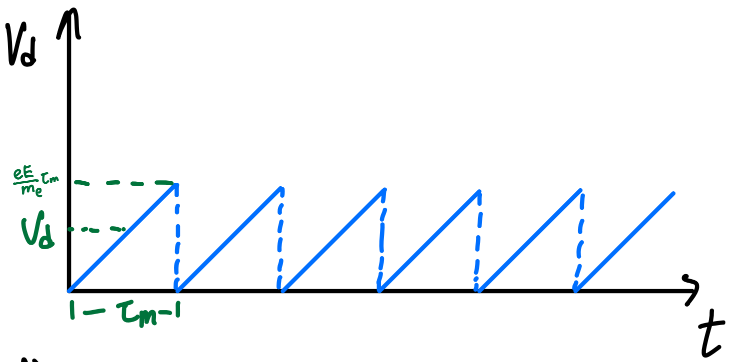
$\rightarrow$ third maximum $x = \frac{3\pi + 4\pi}{2} = \frac{7}{2}\pi$

$$\rightarrow \left(\frac{\sin x}{x}\right)^2 = 0.83\% < 1\% \Rightarrow \underline{\text{third maximum}}$$

④ (a) RMS velocity is $V_{th} = \sqrt{\frac{3k_BT}{m_e}} = 1.17 \times 10^5 \, m/s$

(b) — (e) are about "Drude Theory", but differs by a factor of $\frac{1}{2}$ from normal Drude model.

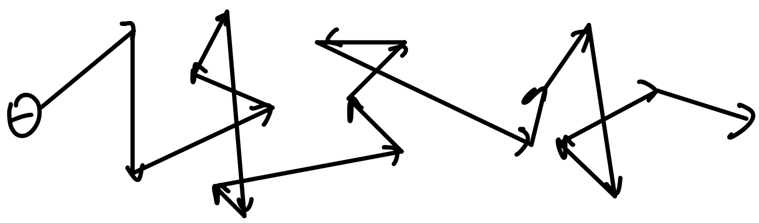
(b) (i)



(ii) $I = nAV_de$

$$\Rightarrow V_d = \frac{I}{nAe} = \frac{1A}{(2.9\times10^{28})(1\times10^{-6})(1.6\times10^{-19})} = 2.16\times10^{-4} \, m/s$$

(iii)



(iv) $X_d = V_d \tau_m$

(c) Resistivity $\underline{E} = \rho \underline{j}$ → current density ; $V_d = \frac{eE}{2m_e}\tau_m$

E-field resistivity

$$j = nV_de \simeq nV_{th}e = \frac{ne^2E\tau_m}{2m_e} \Rightarrow \cancel{E} = \rho\frac{ne^2\cancel{E}\tau_m}{2m_e}$$

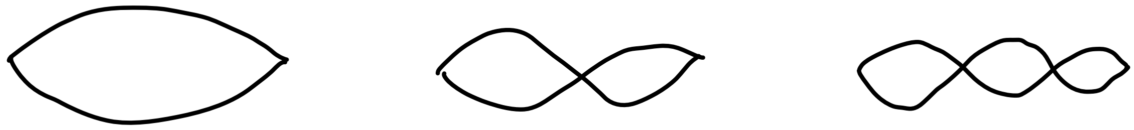
$$\Rightarrow \rho = \frac{2m_e}{ne^2\tau_m} \; ; \; \lambda = V_{th}\tau_m \Rightarrow \rho = \frac{2m_eV_{th}}{ne^2\lambda} = \frac{2m_e}{ne^2\lambda}\sqrt{\frac{3k_BT}{m_e}}$$

(d) At room temperature $\rho(T) = \rho_0 [1 + \alpha (T - T_0)]$ ($\sim T$, not $\sim T^{\frac{1}{2}}$)

(e) $\lambda = \frac{2 m_e V_{th}}{n e^2} \frac{1}{\rho} = \frac{2 m_e V_{th} \sigma}{n e^2} = \underline{2.5 \times 10^{-8} \, m}$

(f) – (k) are about Sommerfeld Free electron Theory

(f)



n = 1 n = 2 n = 3

$$\underline{l = n \frac{\lambda}{2}}$$

(g) $l = n_x \frac{\lambda_x}{2}$, $l = n_y \frac{\lambda_y}{2}$, $l = n_z \frac{\lambda_z}{2}$, $k_x = \frac{2\pi}{\lambda_x}$, $k_y = \frac{2\pi}{\lambda_y}$, $k_z = \frac{2\pi}{\lambda_z}$

wave vector $\underline{k} = (k_x, k_y, k_z)$, $\lambda = \frac{2l}{n}$, $\frac{2\pi}{\lambda} = \frac{2\pi}{2l} \cdot n = \frac{\pi n}{l}$

$$\Rightarrow \underline{k} = \left(\frac{\pi n_x}{l}, \frac{\pi n_y}{l}, \frac{\pi n_z}{l} \right)$$

momentum $\underline{p} = \hbar \underline{k} = \frac{h}{2\pi} \underline{k}$

$$E_F = \frac{p^2}{2m_e} = \frac{h^2 k^2}{8\pi^2 m_e} = \frac{h^2}{8\pi^2 m_e} \cdot \frac{\pi^2}{l^2} (n_x^2 + n_y^2 + n_z^2) = \underline{\frac{h^2}{8 l^2 m_e} (n_x^2 + n_y^2 + n_z^2)}$$

as required.

(h) Pauli exclusion principle ⇒ only 2 electrons in a momentum state

⇒ 2L states in total ⇒ N = 2L

$$N = \frac{1}{4} \cdot \frac{4\pi n_{max}^3}{3} = \frac{\pi}{3} n_{max}^3 \Rightarrow n_{max}^2 = \left(\frac{3N}{\pi} \right)^{\frac{2}{3}}$$

$\because E_F = \frac{h^2}{8 l^2 m_e} n_{max}^2$, use $l^2 = V^{\frac{2}{3}}$ $\Rightarrow \left(\frac{n_{max}}{l} \right)^2 = \left(\frac{3N}{\pi V} \right)^{\frac{2}{3}}$

$$\Rightarrow \underline{E_F = \frac{h^2}{8 m_e} \left(\frac{3N}{\pi V} \right)^{\frac{2}{3}}} = \frac{\hbar^2}{2 m_e} (3\pi^2 n)^{2/3}$$

(classic result)

(i) $E_F = \frac{h^2}{8m_e}\left(\frac{3}{\pi}n\right)^{\frac{2}{3}} = 5.52 \times 10^{-19} J = \underline{3.45 eV}$

(j) (i) $V_F = \sqrt{\frac{2E_F}{m_e}} = 1.1 \times 10^6 \, m/s$

For τ_m, we assume it is the same as the τ_m we had in Drude theory.

$$\tau_m = \frac{2.5 \times 10^{-8}}{1.17 \times 10^5} = 2.14 \times 10^{-13} s$$

$$\Rightarrow \quad \lambda = V_F \tau_m = \underline{2.4 \times 10^{-7} m}$$

(ii) Molar volume $= \frac{M}{\rho_m} = \frac{23 \times 10^{-3}}{0.971 \times 10^3} = 2.37 \times 10^{-5} \, m^3$

Volume per particle $= \frac{2.37 \times 10^{-5} m^3}{N_A} = 3.94 \times 10^{-29} m^3$

$N_A \rightarrow$ Avogadro's Number

average distance between sodium atoms

$$d = (3.94 \times 10^{-29})^{\frac{1}{3}} = 3.4 \times 10^{-10} m = 3.4 \, A^\circ$$

$\frac{\lambda}{d} \simeq \underline{700}$ too many atoms in 1λ, why don't the electrons scatter off the sodium atoms?

$\rightarrow$ solution : Bloch's theorem.

(k) if n = number density :

$\because \; E_F = \frac{\hbar}{2m_e}(3\pi^2 n)^{2/3} \Rightarrow E_F \cdot n^{-\frac{2}{3}} = const \Rightarrow \ln E_F - \frac{2}{3}\ln n = const$

$$\Rightarrow \frac{dE_F}{E_F} - \frac{2}{3}\frac{dn}{n} = 0 \quad \Rightarrow \underline{\frac{dn}{dE_F} = \frac{3N}{2E_F}}$$

if $n =$ energy (momentum) level number $= n_{max}$

$\because \quad E_F = \frac{h^2}{8l^2 m_e} n^2 \Rightarrow E_F n^{-2} = \text{const} \Rightarrow \ln E_F - 2\ln n = \text{const}$

$$\Rightarrow \frac{dE_F}{E_F} - 2\frac{dn}{n} = 0 \Rightarrow \frac{dn}{dE_F} = \frac{n}{2E_F}$$

Assume $\Delta E = \left(\frac{dE_F}{dn}\right)\Delta n$ at $E = E_F$, for 1 level difference,

say $\Delta n = 1 \Rightarrow \Delta E = \frac{dE_F}{dn} = \frac{2E_F}{n} = \frac{h^2 n}{8l^2 m_e}$

$\because n = n_{max} = \left(\frac{3N}{\pi}\right)^{\frac{1}{3}}, \quad l = V^{\frac{1}{3}}$

$$\Rightarrow \Delta E = \frac{h^2}{8lm_e}\left(\frac{n}{l}\right) = \frac{h^2}{8lme}\left(\frac{3N}{\pi V}\right)^{\frac{1}{3}} = \frac{h^2}{8lm_e}\left(\frac{3n}{\pi}\right)^{\frac{1}{3}}$$

number density, not level number

$$= (1.82 \times 10^{-28}\ \mathrm{J \cdot m}) \times \frac{1}{l} = (1.14 \times 10^{-9}\ \mathrm{eV \cdot m}) \times \frac{1}{l}$$

small for normal sized metal blocks.

$\Rightarrow$ this makes sense because $k \sim \frac{n\pi}{l}$, large l gives smaller Δk and hence smaller ΔE