BPhO 2020 Round 1
Section 2 Question 5

Ziyan Li

Solutions

(a) As shown in the figure, we first want to know the radius $r = |OA|$ of the circular path that the car travels through. This is given by the so called *intersecting chords theorem* which was given in the question description as a hint. This theorem states that $|AS||SB| = |CS||SD|$, which can be easily proven by connecting AD and BC and using the fact that $\angle ADS = \angle SBC = \frac{1}{2}\angle AOC$ for a circle and hence $\triangle ADS \sim \triangle CBS$. Note that AD does not need to be perpendicular to BC for this theorem to hold.

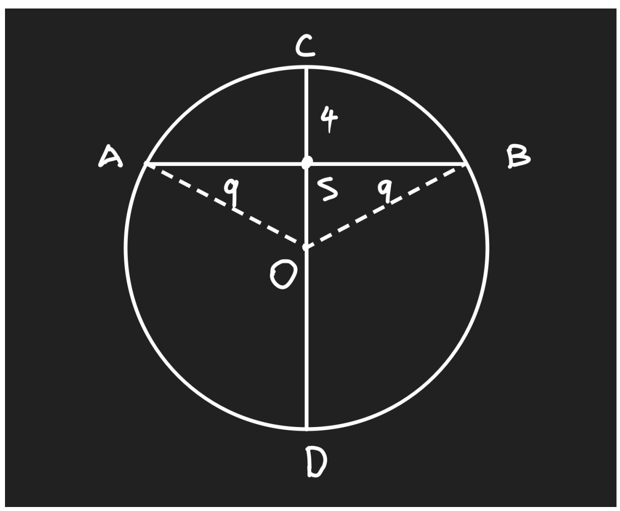

Figure 1: (a)

From information given in the question, $|AS| = |SB| = \frac{1}{2}|AB| = \frac{18}{2} = 9$, and $|CS| = 4$, so we have $|SD| = \frac{81}{4}$. CD is the diameter so $|CD| = 2r = \frac{81}{4} + 4 = \frac{97}{4}$. So

$$r = \frac{97}{8} = 12.125\,\text{m} \tag{1}$$

At the top of the bridge, when speed is maximum there is no normal force. The weight of the car alone provides the centripetal acceleration. So

$$mg = m\frac{v^2}{r} \tag{2}$$

and we have

$$v = \sqrt{gr} = 10.9\,\text{m/s} \tag{3}$$

(b)

i Assuming circular orbit and mass of the Sun to be m, we have

$$\frac{GM_g m}{r^2} = m\omega^2 r = m(\frac{4\pi^2}{T^2})r \tag{4}$$

so that

$$T = \frac{2\pi}{\sqrt{GM_g}} r^{3/2} \tag{5}$$

ii Rearranging the above gives

$$M_g = \frac{4\pi^2 r^3}{GT^2} \tag{6}$$

Substitute in values gives $M_g = 3.4 \times 10^{41}$ kg.

iii Take numerical values

$$N = \frac{M_g}{M_s} = 1.7 \times 10^{11} \tag{7}$$

stars like the Sun.

(c)

i For a satellite of mass m

$$\frac{GM_E m}{r^2} = m\frac{v^2}{R} \tag{8}$$

So

$$v = \sqrt{\frac{GM_E}{r}} \tag{9}$$

ii Grazing the Earth meaning we can used the approximation $r \approx R_E$ so

$$v = \sqrt{\frac{GM_E}{R_E}} = \sqrt{\frac{GM_E}{R_E^2} R_E} = \sqrt{gR_E} = 7.9\,\text{km/s} \tag{10}$$

This is known as the first cosmic speed.

iii

$$T = \frac{2\pi R_E}{v} = \frac{2\pi}{\sqrt{GM_E}} R_E^{3/2} = 5060\,\text{s} \tag{11}$$

iv As shown before, period for any r around the Earth is

$$T(r) = \frac{2\pi}{\sqrt{GM_E}} r^{3/2} \tag{12}$$

Take logarithm of both sides gives

$$\ln T = \ln \frac{2\pi}{\sqrt{GM_E}} + \frac{3}{2} \ln r \tag{13}$$

Differentiate both sides with respect to r gives

$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{3}{2} \cdot \frac{1}{r} \tag{14}$$

So at around $r \approx R_E$, $\mathrm{d}r = \Delta r = h$, $\mathrm{d}T = \Delta T$, we have

$$\frac{\Delta T}{T} = \frac{3}{2} \cdot \frac{h}{R_E} \tag{15}$$

The new period is

$$T' = T(1 + \frac{\Delta T}{T}) = T(1 + \frac{3h}{2R_E}) = 5300\,\mathrm{s} \tag{16}$$

v The graph is as shown. The relationship is linear.

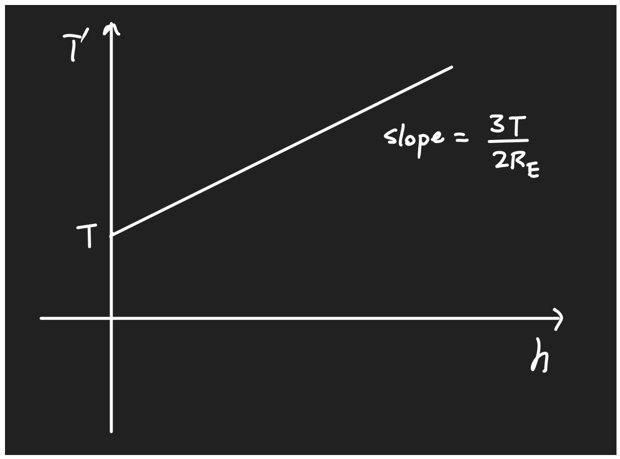

Figure 2: (b)(v)

vi For this elastic collision, we write down the conservation of momentum equation and the relative velocity equation representing the conservation of energy

$$\begin{aligned} m_1 u &= m_1 v_1 + m_2 v_2 \\ u - 0 &= -(v_1 - v_2) \end{aligned} \tag{17}$$

Solve for v_2 gives

$$v_2 = \frac{2m_1}{m_1 + m_2} u \approx 2u \tag{18}$$

where in the last step we have used the condition $m_1 \gg m_2$. This can also be viewed in the reference frame of m_1. In this frame it looks like the much smaller mass m_2 collides elastically onto a wall with velocity $-u$, where the minus sign indicates the direction, so it's velocity should be reversed to $+u$ in this frame after collision. And transferring back to the lab frame gives an additional relative velocity u so $v_2 \approx 2u$.

The kinetic energy lost by m_1 is simply the kinetic gained by m_2 because this is an elastic collision. This is

$$K_2 = \frac{1}{2} m_2 v_2^2 \approx 2m_2 u^2 \tag{19}$$

vii In time interval Δt, the satellite sweeps a volume $\Delta V = Av\Delta t$ in space. In this volume, it collides with $\Delta N = n\Delta V$ number of particles, where n is the number density, defined as the number of particles per volume. If each gas particle has a mass of m, then according to the result in the previous question, each collision by a gas particle reduces the kinetic energy of the satellite by $2mv^2$. So the total change in kinetic energy of the satellite is $\Delta K = 2mv^2\Delta N = 2mv^2 n\Delta V = 2mv^3 nA\Delta t$. Since the mass density ρ is related to the number density n by $\rho = mn$, divide through we have the final expression

$$\frac{\Delta K}{\Delta t} = 2\rho A v^3 \tag{20}$$

viii We will constantly use d and Δ interchangibly. Since $K = \frac{1}{2}mv^2$, we have

$$\frac{\mathrm{d}K}{\mathrm{d}t} = mv\frac{\mathrm{d}v}{\mathrm{d}t} \tag{21}$$

so

$$\Delta K = mv\Delta v \tag{22}$$

and we have

$$\frac{\Delta v}{v} = \frac{\Delta K}{mv^2} \tag{23}$$

Using $v = 7.9$ km/s and other numerical values gives $\frac{\Delta v}{v} = 8 \times 10^{-6}$.

ix The total energy of the satellite is the sum of kinetic and potential energies

$$E = K + U = \frac{1}{2}m(\sqrt{\frac{GM_E m}{r}})^2 - \frac{GM_E m}{r} = -\frac{GM_E m}{2r} = -K \tag{24}$$

So when total energy decreases, the kinetic energy increases and thus the speed increases.

This may seem to be a very weird conclusion, because the collisions seem to reduce the kinetic energy directly, so why do we take the detour to consider the total energy first and then track the logic back to kinetic energy to get a different result?

Note we assumed that the changes in orbital velocity and orbital radius are so small that the orbit in any revolution can effectively be regarded as a circular orbit. But what really happens is that the orbits are never perfectly circular. Because the decrease in energy due to collisions is so small, we can think about the situation as one bigger collision per half-revolution instead of many continuous collisions every second.

At the instant when a collision reduces the speed of the satellite, it actually goes into a elliptical orbit. The total energy is reduced so the position of the collision is actually the aphelion of the new elliptical orbit. At this moment of collision the speed actually decreases, but as the satellite starts its elliptical motion the distance from the centre of the Earth to the satellite decreases as well until the satellite reaches the perihelion. In this process the speed increases back because the potential energy decreases and gravitational force does positive work. At this perihelion the satellite encounters a new collision and thereby decreases its speed again, but this perihelion becomes the aphelion of a new elliptical orbit and as the process continues, the satellite becomes closer and closer to the Earth and its speed becomes faster and faster.

In short, collision decreases the speed of the satellite, but due to the collision, gravity can increase the speed of the satellite back more.

x We can apply directly the equation we obtained before to get

$$\rho = \frac{\mathrm{d}K/\mathrm{d}t}{2Av^3} = 8.4 \times 10^{-13}\,\mathrm{kg/m^{-3}} \tag{25}$$

(d)

i As shown in the figure, the star with mass km orbits from A to A', and the star with mass m orbits from B to B'. The centre of mass of the system is O. We let $|AB| = |A'B'| = r_0$

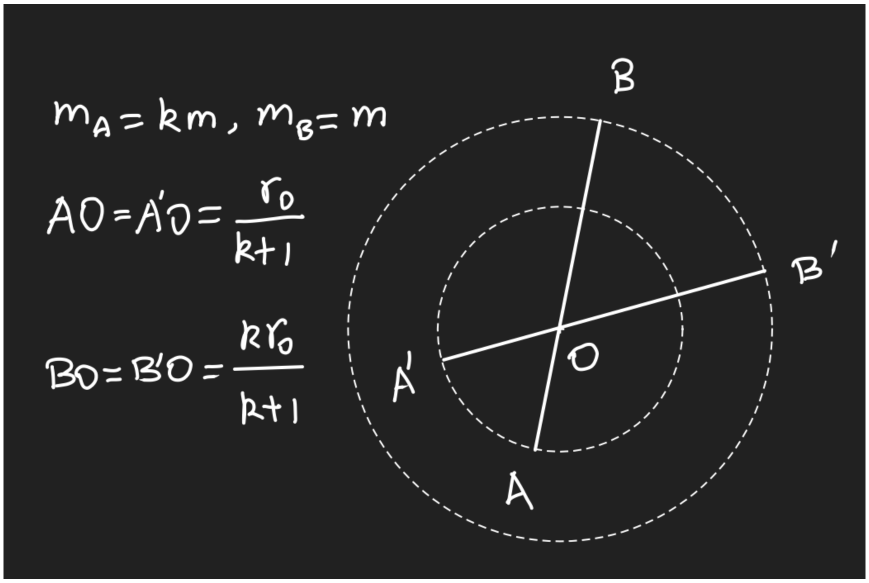

Figure 3: (d)(i)

ii Relative orbital radii

$$\frac{|AO|}{|BO|} = \frac{1}{k} \tag{26}$$

and their absolute values in terms of r_0 are shown in the figure.

iii The qualitative description of the subsequent motion is that the new centre of mass of the two stars gains a constant velocity and the motion of the two stars relative to their new centre of mass is an ellipse.

Here we try to quantitatively describe the motion of this binary system in detail. First we notice that when the mass of km suddenly changes to m, the centre of mass of the system shifts to the midpoint of the two stars O' instead of the original O.

Let the original velocities of the two stars are v and kv. Since the velocities of the two stars do not change at the moment of explosion, the CM velocity after the explosion is

$$v_c = \frac{m(kv) + m(-v)}{2m} = \frac{k-1}{2}v \tag{27}$$

towards the direction of motion of the lighter star at the moment of the explosion. This non-zero CM velocity of the two stars after explosion is consistent with the conservation of momentum because a spherically symmetric explosion in the frame of the heavier star km will cause the CM of the remnants of the exploded star to have a non-zero momentum relative to the lab frame.

We need now to calculate the subsequent relative motion of the two stars. This requires the knowledge of the so called *reduced mass*.

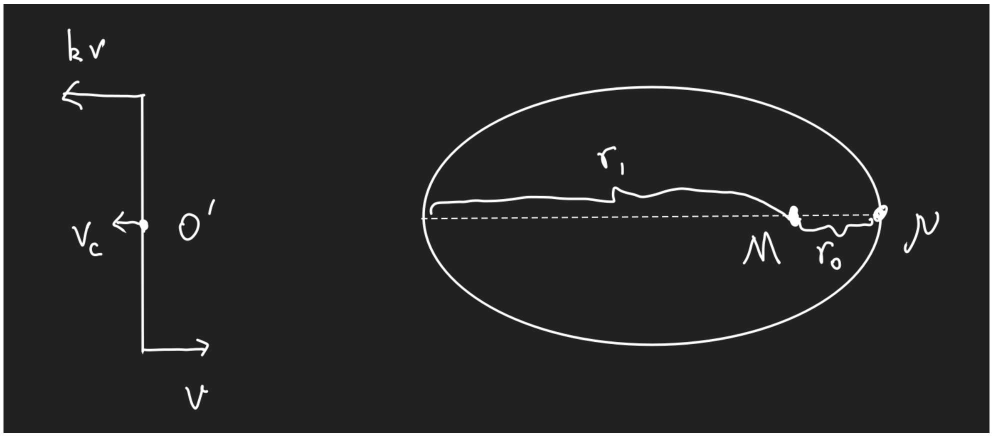

Figure 4: (d)(iii)

Before collision the reduced mass is

$$\mu = \frac{(km)(m)}{km+m} = \frac{km}{k+1} \tag{28}$$

The effective central star has mass

$$M = \frac{(km)(m)}{\mu} = (k+1)m \tag{29}$$

which is the sum of masses.

Let v_0 be the relative velocity between km and m, and r_0 be their relative distance. We have the Newton's Second Law for the relative motion

$$\frac{GM\mu}{r_0^2} = \mu\frac{v_0^2}{r_0} \tag{30}$$

so

$$v_0 = \sqrt{\frac{GM}{r_0}} = \sqrt{\frac{G(k+1)m}{r_0}} \tag{31}$$

Note that $v_0 = (k+1)v$, so

$$v_c = \frac{k-1}{2(k+1)}v_0 \tag{32}$$

After the explosion, the reduced mass changed to

$$\mu' = \frac{m^2}{m+m} = \frac{m}{2} \tag{33}$$

and the effective central star has mass

$$M' = \frac{m^2}{\mu'} = 2m \tag{34}$$

By theories about the reduced mass, the motion of μ under the gravitational field of a fixed star with mass M describes the relative motion between the binary stars before the explosion, and the motion of μ' under the gravitational field of a fixed star with mass M' describes the relative motion between the binary stars after the explosion.

After the explosion, the relative motion can be regarded as a planet with mass μ' orbiting a central star with mass M' in an elliptical orbit as shown in the figure. The velocity of μ' right after explosion is v_0, this is not affected by the explosion. The position of explosion is the perihelion as we will later prove, but now we know for sure it is one of the apsides because the velocity is perpendicular to the radial position at this point (as in the previous circular orbit), and only perihelion or aphelion satisfies this condition.

In this setup, we denote the distance from the aphelion (or the other apside as for now) to the central star to be r_1 and the velocity at the aphelion to be v_1, so the conservation of energy

$$E = \frac{1}{2}\mu' v_0^2 - \frac{GM'\mu'}{r_0} = \frac{1}{2}\mu' v_1^2 - \frac{GM'\mu'}{r_1} \tag{35}$$

and the conservation of angular momentum

$$L = \mu' v_0 r_0 = \mu' v_1 r_1 \tag{36}$$

So we have

$$v_1 = \frac{r_0}{r_1} v_0 = \frac{r_0}{r_1}\sqrt{\frac{GM}{r_0}} = \sqrt{\frac{GMr_0}{r_1^2}} \tag{37}$$

and substitute this back to the energy equation gives

$$\frac{G\mu'}{r_0}\left(\frac{M}{2} - M'\right) = \frac{1}{2}\mu' \frac{GMr_0}{r_1^2} - \frac{GM'\mu'}{r_1} \tag{38}$$

which gives us

$$\frac{Mr_0}{2}\left(\frac{1}{r_1}\right)^2 - M'\left(\frac{1}{r_1}\right) + \frac{1}{r_0}\left(M' - \frac{M}{2}\right) = 0 \tag{39}$$

This is a quadratic equation for $\frac{1}{r_1}$, with one trivial solution being $r_1 = r_0$, so the other non trivial one can be easily obtained using the Vieta's theorem:

$$\frac{1}{r_1} + \frac{1}{r_0} = \frac{-(-M')}{Mr_0/2} \tag{40}$$

so

$$r_1 = \frac{M}{2M' - M} r_0 = \frac{k+1}{2 \cdot 2 - (k+1)} r_0 = \frac{k+1}{3-k} r_0 \tag{41}$$

and since $k > 1$, we must have $M' < M$ and $r_1 > r_0$ so indeed r_0 is the perihelion and r_1 is the aphelion. Hence the semi-major axis of the ellipse is

$$a = \frac{r_0 + r_1}{2} = \frac{2}{3-k} r_0 \tag{42}$$

Therefore, after the explosion, the relative motion of the binary stars undergoes an ellipse with semi-major axis being $\frac{2}{3-k}$ times the initial separation of the two stars before explosion, and the centre of mass has a velocity $\frac{k-1}{2(k+1)}$ times the initial relative velocity between the stars before explosion.

iv As we can see from the previous question, as $k \geq 3$ the semi-major axis blows up, so the orbit becomes unbounded. And we require $k > 1$ from the beginning. So overall we need

$$1 < k < 3 \tag{43}$$

for the orbit to remain bounded.

This result can also be calculated directly using energy considerations. As the explosion does not change velocities in the deep space lab frame, the kinetic energies right after explosion is the same as before, but we have to be careful here because part of this kinetic energy after explosion becomes the centre of mass kinetic energy and this does not go away even if the stars are separated to infinity. The condition for the orbit to be just unbounded is the kinetic energy in the CM frame plus the potential energy equals zero.

In CM frame after explosion the two stars have the same mass thus same velocities v'. This has the expression

$$v' = kv - v_c = v_c + v = \frac{k+1}{2}v = \frac{(k+1)}{2(k+1)}v_0 = \frac{v_0}{2} = \frac{1}{2}\sqrt{\frac{G(k+1)m}{r_0}} \tag{44}$$

Kinetic energy plus potential energy in CM frame less than or equal to zero gives the condition for bounded orbit

$$2 \times \frac{1}{2}m\left(\frac{1}{2}\sqrt{\frac{G(k+1)m}{r_0}}\right)^2 - \frac{Gm^2}{r_0} < 0 \tag{45}$$

so

$$\left(\frac{k+1}{4} - 1\right)\frac{Gm^2}{r_0} < 0 \tag{46}$$

which gives $k + 1 < 4$ and thus

$$1 < k < 3 \tag{47}$$

so recovers the results obtained from analysing the orbit. If curious about a more general case in which the mass of the heavier star after explosion is not the same as the lighter star, please refer to the following literature , in Chinese of course.

Figure 5: (d)(iv)