First Year Physics: Prelims CP1

Classical Mechanics: Prof. Neville Harnew

Problem Set II: Applications of the Equation of Motion

Questions 1-10 are "standard" examples. Questions 11-16 are additional questions that may also be attempted or left for revision. Problems with asterisks are either more advanced than average or require extensive algebra. All topics are covered in lectures 6-9 of Michaelmas term.

Standard Questions

Work and Conservation of Energy

1. The Work Energy Theorem:

(a) Show that the work done by the force $F(x)$ to move a particle of mass m from a to b is given by: $W_{ab} = T(b) - T(a)$ where $T(x)$ is the kinetic energy at the point x. Does this only apply for conservative forces?

(b) The particle moves towards the origin of the x-axis under the influence of the force $F = k/x^2$ where k is a positive constant. If the velocity at $x = x_\circ$ is u, use the result in (a) to find the distance of minimum approach of the particle to the origin.

2. Gravitational potential of two point masses: A particle P of mass m moves on the x-axis under the combined gravitational attraction of two particle, each of mass M, fixed at the points $(0, \pm a)$ in the $x - y$ plane.

(a) Show that the force acting on the particle P is given by

$$F(x) = -\frac{2mMGx}{(a^2 + x^2)^{3/2}}$$

where G is the gravitation constant.

(b) Find the corresponding potential energy. Discuss the possible motion of the particle using the potential energy curve.

(c)Assuming that P is initially released from rest at $x = 3a/4$, find the maximum speed achieved by the particle in the subsequent motion.

Projectiles in 2D

3. A ball is thrown with initial speed V, at an angle θ above the horizontal, over flat ground. Show that the motion is restricted to a single plane. Neglecting air resistance, find

(a) the time taken for the ball to reach the ground;

(b) the maximum height reached by the ball;

(c) the horizontal distance travelled by the ball before hitting the ground (range);

(d) the kinetic and potential energies at the position of maximum height (and verify that their sum equals the initial energy).

The following are experimental data on the range and muzzle velocity of mortar shells, all fired at 45° to the horizontal. The time of flight is also tabulated. Compare the ranges and times with the simple theory above. How would you explain any differences?

Muzzle Velocity (m/s)	Range (m)	Time (s)
101.8	972	14.4
112.2	1160	15.7
121.9	1349	17.0
131.4	1539	18.2

Resistive Forces

4. A body of mass 1 kg falls from a height of 100 m and reaches the ground with a speed of 40 m s^{-1}. What is the average air resistance? Discuss the meaning of the term 'average' in this connection.

5. An air-filled toy balloon, with a diameter of 30 cm and a mass (not counting the air inside) of about 0.5 g, falls from rest in air.

(a) *Estimate* the terminal speed of fall, assuming a linear drag law [i.e. assume the balloon is a sphere, and that Stokes' formula for viscous drag is applicable: drag force $F_1 = 6\pi a\eta v$ [N], where a is the radius of the sphere [m], $\eta = 1.7 \times 10^{-5}$ kg m^{-1} s^{-1} is the viscosity of air and v is velocity].

(b) Given this drag law, show that the downward velocity v of the balloon satisfies a differential equation of the form

$$\frac{dv}{dt} = g - \lambda v \ ,$$

and give an expression for the quantity λ. Solve this differential equation, using the appropriate initial condition, and hence find how long it takes for the balloon to come to within 5 per cent of its terminal speed.

(c) How would your result for (a) change if an additional drag force due to turbulent eddies of the form $F_2 = 0.87(av)^2$ [N] (where a is in m and v is in m s^{-1}) were taken into account? Which is more realistic?

6. A ball of mass m is projected vertically upwards at a velocity $v_\circ$. The ball experiences an air resistance force (in addition to gravity) of the form $F_R = -\alpha v^2$ where $\alpha > 0$ is constant and v is the velocity, and reaches a maximum hight h before it returns back to the point of projection.

(a) Write down the equations of motion of the ball during its upward and and downward journeys and show that the maximum height reached is given by

$$h = a \ln[1 + (v_\circ/v_l)^2]$$

where $v_l = \sqrt{(mg/\alpha)}$ and $a = (v_l)^2/(2g)$.

(b) Show that the velocity of the ball when it returns back to the point of projection is given by

$$v_r^2 = v_l^2[1 - \exp(-h/a)].$$

Motion of Rockets

7. Two masses m and δm ($\delta m \ll m$) are joined together, and moving with velocity v in some inertial frame. At some instant they fly apart along their direction of motion with relative velocity V (e.g., because of the release of a spring), such that the velocity of the mass m becomes $v + \delta v$. Show that, if the small mass is projected in the backwards direction,

$$m\delta v = V\delta m \ .$$

Explain carefully whether the same expression applies for the case in which the small mass is projected forwards.

We now specialize to the case of a rocket which burns fuel, so that the mass of the rocket remaining at time t is $m(t)$. Show that the acceleration of the rocket is given by

$$\frac{dv}{dt} = -\frac{V}{m}\frac{dm}{dt} \ .$$

Explain why this expression has a minus sign, while the one in (a) above does not. Finally, modify the rocket equation to take account of a constant gravitational field (acceleration due to gravity = g).

8. A rocket of initial mass M, of which half is fuel, is launched vertically upwards at $t = 0$. It burns the fuel at a mass rate of α (a positive quantity) and ejects it backwards from the rocket at a velocity V with respect to the rocket. (a) Show that the equation governing the (vertical) speed v of the rocket is

$$\frac{dv}{dt} = -g + \frac{\alpha V}{M - \alpha t}$$

where g is the acceleration due to gravity, assumed constant. (b) Show that the rocket cannot leave the launch-pad on ignition unless $\alpha V > Mg$. (c) Given $\alpha V > Mg$, show that fuel burn-out occurs at $t = M/2\alpha$. (d) Show that the upward velocity at burn-out is $v = V \ln 2 - gM/(2\alpha)$ (e) Find the height of the rocket at burn-out and the maximum height reached.
[You need $\int \ln x \, dx = x \ln x - x$.]
[Ans for (e) max. height is $(1 - 2\ln 2)(MV/2\alpha) + (V \ln 2)^2/(2g)$]

9.* **Two stage rocket:** A rocket of total initial mass (casing + fuel + payload) Nm, where m is the payload mass, ejects mass at a constant velocity u relative to the rocket. The ratio of (casing mass) to (casing + fuel mass) is r. Ignore gravity and air-resistance and assume that the rocket starts from rest.

(i) One stage rocket. Show that the payload achieves a final velocity of:

$$v_1 = u \ln\left[\frac{N}{rN + (1 - r)}\right].$$

(ii) Two stage rocket. The total mass of both stages (fuel+casing+payload) is now Nm and the total mass of the second stage (fuel+casing+payload) is nm. The other conditions for the second stage, including those for u and r, are the same as for the first stage in part (i). Assuming that the first stage is dropped when its fuel is exhausted show that the final velocity of the payload is:

$$v_2 = u \ln\left[\frac{N}{rN + n(1 - r)}\right] + u \ln\left[\frac{n}{rn + (1 - r)}\right].$$

(iii) If N and r are constant, show that v_2 is a maximum for $n^2 = N$ and then:

$$v_2^{max} = 2u \ln\left[\frac{n}{rn + (1 - r)}\right].$$

(iv) Realistic maximum values of u from chemical combustion are of the order of $2.9\,\mathrm{kms}^{-1}$. If $r = 0.1$ could Earth escape speed be reached with either a one or two stage rocket?

Non-inertial reference frames

10. A passenger holding a parcel of mass m is standing in a lift which is being accelerated upward by a constant force F. The total mass of lift plus passenger is M. (Assume $M >> m$.)

(a) What is the acceleration of the lift?

(b) What weight does the parcel appear to have for the passenger?

(c) If the passenger drops the parcel from height h, how long does it take to reach the floor? Try this both in the lift frame (non-inertial) and an inertial frame (eg w.r.t. lift shaft).

(d) If the lift changes to uniform velocity, what must the value of F become? Comment on your result.

Additional questions

11.* Gravitational Potential of the Earth Denote the radius of the Earth by R_E and the value of the acceleration due to gravity at the surface by g.

(a) Show that free-fall in the Earth's gravitational field from infinity results in the same velocity at the surface as if the object had dropped a distance R_E under a constant acceleration of $g = GM_E/R_E^2$, where G is the gravitational constant and M_E is the mass of Earth.

(b) Show that for heights h small compared to R_E the first correction to the velocity at the surface for an object in free-fall is: $v = \sqrt{2gh}\left(1 - \frac{h}{2R_E}\right)$.

(c) For a height h small compared to R_E show that the fractional change in period of a pendulum clock is given by: $\Delta T/T = h/R_E$. Does the clock gain or lose and evaluate the change for a height of 100 m in seconds per week?

12.* A canon is placed on an inclined plane of angle α with the horizontal. The gun is aligned with the steepest line of the plane and can fire shells in both the upward and downward directions, at an initial speed V and an angle θ with respect to the plane (upwards).

(a) Find where the shell will land up the plane for $\theta = \alpha = 30^\circ$ (Ans: (a) $\frac{2}{3}(V^2/g)$)

(b) Show that the ratio of the *maximum* ranges up and down the plane is given by:

$$\frac{R_{up}}{R_{down}} = \frac{1 - \sin\alpha}{1 + \sin\alpha}$$

13. A body of mass M, travelling in a straight horizontal line, is supplied with constant power P and is subjected to a resistance Mkv^2, where v is its speed and k is a constant.

(a) Show that the speed of the body cannot exceed a certain value v_m and find an expression for v_m.

(b) Show that, starting from rest, the body acquires half the maximum speed after travelling a distance $\ln(8/7)/(3k)$.

(c) If the power is then cut off and an additional retarding force of constant value F imposed, find the subsequent time which elapses before the body comes to rest.

[Ans: (a) $v_m^3 = P/(Mk)$; (c) time= $\sqrt{M/Fk}\tan^{-1}[0.5v_m\sqrt{Mk/F}]$.]

14.* Work against non-conservative forces

This question follows on from Question 6:

(a) Calculate by direct integration the energy dissipation during the upward journey and show that your result is consistent with the work-energy theorem.

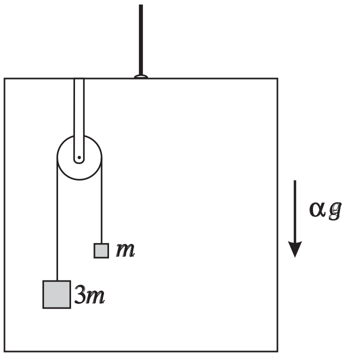
(b) Using the result of Q6 (b) for the final velocity, show that for $v_\circ = v_l$ half of the initial energy is lost during the round trip.

15. Suppose that he two masses in the two-body problem of set I (Q6) are subject to an external force which is proportional to the mass of the particle on which it acts, i.e., we are in a uniform gravitational field, so that we can set $F_i = m_i g$ $(i = 1, 2)$.

(a) From your result for 6(d), demonstrate that the acceleration of the centre of mass is simply g.

(b) Still within this special case, consider the motion of the masses relative to the centre of mass. Guided by the calculation in 6(d), show that the equations of motion of the masses relative to their centre of mass are also given by Newton's second law including only the internal forces.

16.* A lift has a downward acceleration αg ($\alpha \leq 1$). Inside the lift is mounted a pulley, of negligible friction and inertia, over which passes an inextensible string carrying two objects of masses m and $3m$ (see below).



We wish to find (i) the acceleration of the object of mass $3m$ with respect to the lift, and (ii) the force exerted on the pulley by the rod that joins it to the roof of the lift. Use two frames to analyse the motion: one (inertial) fixed in the lift shaft; the other (non-inertial) fixed in the lift. Denote the coordinates of the masses m and $3m$ with respect to the inertial reference frame by x_1 and x_3 and with respect to the non-inertial frame by x_1' and x_3' respectively.

(a) Let the tension in the string be T and the accelerations of the masses m and $3m$ be a_1 and a_3 respectively (in the inertial frame). Write down the equations of motion (i.e., the expressions of Newton's second law) for each of the two masses in terms of T, in the inertial frame. Would these equations be different if the lift were not accelerating?

(b) Eliminate T from your equations, and hence relate the accelerations a_1 and a_3.

(c) Write down another relation between a_1 and a_3 by setting up equations relating the motion in the inertial (shaft) frame to the non-inertial (lift) frame and assuming that the string is inextensible.

(d) You should now have enough information to solve the problem in either frame. Give answers for the cases $\alpha = 1/3$ and $\alpha = 1$, what is special about the second case?

(e) Could one get all the above results by assuming the lift not to be accelerating, but taking the acceleration due to gravity to be $g(1 - \alpha)$?

[Ans: for the case $\alpha = 1/3$, accl. of mass $3m$ in lift frame $2g/3$, force on pulley $2mg$.]