

Mechanics of Ziyan Li

1 (a)

According to Fermat's Principle, light travels through the trajectory that takes it the ~~shortest~~ shortest, or longest (i.e. stationary) time

When light travels through a distance ds the time taken is $dt = \frac{ds}{V}$, where $V = V(y) = \frac{c}{n(y)}$ is the speed of light as a function of y .

To find the trajectory we need to find the minimize the functional

~~Say~~ time taken from (x_0, y_0) to (x_1, y_1)

$$T = S(y) = \int_{x_0}^{x_1} \frac{ds}{V}$$

$$\therefore V = \frac{c}{n(y)} \quad \therefore \frac{1}{V} = \frac{n(y)}{c} = c^{-1} n(y)$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + y'^2} dx$$

$$\therefore S(y) = c^{-1} \int_{x_0}^{x_1} n(y) \sqrt{1 + y'^2} dx \quad \text{is}$$

the functional we need to minimize

(b) In this case light stays in air so
 $n(y) = 1$ for all y except at boundary.

$$\therefore S(y) = c^{-1} \int_{x_0}^{x_1} n(y) \sqrt{1+y'^2} dx$$

let ~~$F(x)$~~ $F(y, y', x) = \sqrt{1+y'^2} n(y)$

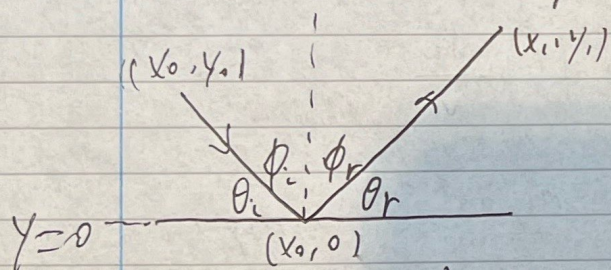
by Euler-Lagrangian Equation

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = \frac{\partial F}{\partial y}$$

$\therefore F$ does not explicitly depend on x

~~$$\therefore \frac{\partial F}{\partial y} = 0 \quad \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$$~~

~~$$\therefore \frac{\partial F}{\partial y'} = C \quad (C \text{ is a constant})$$~~



(Note, F does depend on y because of the boundary)

Consider $\frac{d}{dx} \left(y' \frac{\partial F}{\partial y'} \right) = y' \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + y'' \frac{\partial F}{\partial y'}$

BQ $\therefore \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = \frac{\partial F}{\partial y}$

$$\therefore \frac{d}{dx} \left(y' \frac{\partial F}{\partial y'} \right) = y' \frac{\partial F}{\partial y} + y'' \frac{\partial F}{\partial y'}$$

$\therefore F$ does not explicitly depend on x

$$\therefore F = F(y, y')$$

$$\therefore \frac{dF}{dx} = \frac{\partial F}{\partial y} \frac{dy}{dx} + \frac{\partial F}{\partial y'} \frac{dy'}{dx} = y' \frac{\partial F}{\partial y} + y'' \frac{\partial F}{\partial y'} = \frac{d}{dx} \left(y' \frac{\partial F}{\partial y'} \right)$$

$$\therefore \frac{d}{dx} \left(F - y' \frac{\partial F}{\partial y'} \right) = 0$$

$$\therefore F - y' \frac{\partial F}{\partial y'} = C, \text{ a constant}$$

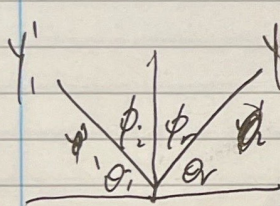
$$\therefore F = n(y) \sqrt{1+y'^2} \quad \text{and} \quad F - y' \frac{\partial F}{\partial y'} = C.$$

$$\therefore n(y) \sqrt{1+y'^2} - \frac{n(y) y'^2}{\sqrt{1+y'^2}} = C$$

$$\therefore \frac{n(y)(1+y'^2)}{\sqrt{1+y'^2}} - \frac{n(y) y'^2}{\sqrt{1+y'^2}} = C$$

$$\therefore \frac{n(y)}{\sqrt{1+y'^2}} = C$$

$$\therefore n(y) = 1 \quad \text{for} \quad y \neq 0 \quad (\text{not at boundary})$$



$$\therefore (1+y'^2)^{1/2} \text{ is constant}$$

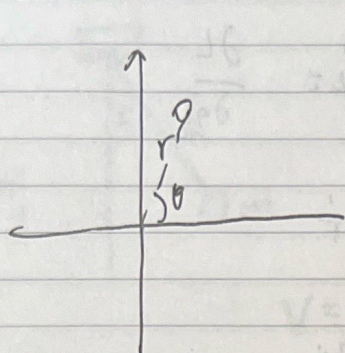
$$\therefore y'^2 \text{ is constant}$$

$\therefore$ the slopes of ~~the~~ incident and reflected light rays are the ~~reciprocal~~ ~~negative~~ negative of each other ($y'_1 = -y'_2$, $y'_2 = -y'_1$).

$$\therefore \tan \theta_i = \tan \theta_r \quad \therefore \theta_i = \theta_r$$

$$\therefore \phi_i = \phi_r \quad \text{as required.}$$

2.



~~(2)~~

velocity $\underline{v} = \dot{r} \hat{r} + r\dot{\theta} \hat{\theta}$

Kinetic Energy $T = \frac{1}{2} m \underline{v} \cdot \underline{v}$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} \Rightarrow m\ddot{r} = -\frac{\partial V}{\partial r} + m\dot{\theta}^2 r$$

~~$$\therefore m\ddot{r} = -\frac{\partial V}{\partial r} + m\dot{\theta}^2 r$$~~

$$\Rightarrow m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r} \quad (1)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt} (mr^2 \dot{\theta}) = 0 \quad (2)$$

$\therefore mr^2 \dot{\theta}$ is a conserved quantity.

In fact, $mr^2 \dot{\theta}$ represents the angular momentum of the particle. $\because V = V(r)$, i.e. the potential energy only depend on r ,

The force acting on the particle is a central force. $\therefore$ Angular momentum must be conserved.

The Hamiltonian

$$H = \sum_k p_k \dot{q}_k - L$$

$$(p_k = \frac{\partial L}{\partial \dot{q}_k})$$

$$\therefore p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\therefore H = (m\dot{r})(\dot{r}) + (mr^2\dot{\theta})(\dot{\theta}) - L$$

$$= m(\dot{r}^2 + r^2\dot{\theta}^2) - \left[\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \right]$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$

θ does not appear in the Hamiltonian

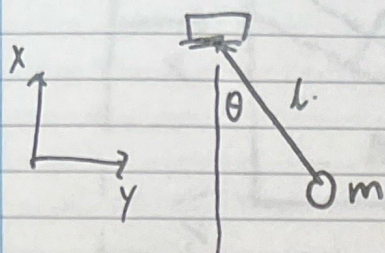
$\therefore \theta$ is a cyclic coordinate.

$$\therefore \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\therefore p_\theta = mr^2\dot{\theta} \text{ is}$$

a conserved quantity.

3 (a)



1 d.o.f.
Generalized coordinates: θ

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (l \dot{\theta})^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$V = mgy = -mgl \cos \theta$$

$$\therefore L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos \theta$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} \quad \therefore m l^2 \ddot{\theta} = -mgl \sin \theta$$

for small θ $\sin \theta \approx \theta$

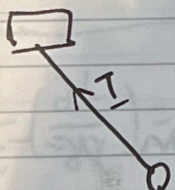
$$\therefore m l^2 \ddot{\theta} = -mgl \theta \quad \therefore \ddot{\theta} + \frac{g}{l} \theta = 0$$

$$\therefore \theta = A \cos(\omega t + \phi) \quad \omega = \sqrt{\frac{g}{l}}$$

$$\therefore \text{Period } P = \frac{2\pi}{\omega} = \boxed{2\pi \sqrt{\frac{l}{g}}}$$

(b) In the ~~horizontal~~ noninertial frame accelerating with the pendulum support vertically.

We add a fictitious force $\underline{F} = -m\underline{a}$ to the bob



In polar coordinates:

$$\underline{F} = -m\underline{a} = m a \cos \theta \underline{\hat{e}}_r - m a \sin \theta \underline{\hat{e}}_\theta$$

$$\begin{aligned} \underline{mg} &= \underline{F}_g \\ \underline{-ma} &= \underline{F} \end{aligned}$$

tension ~~of string~~ of string is $\underline{T} = T \underline{\hat{e}}_r$

Virtual displacement along θ is $\delta \underline{\theta} = \delta \theta \underline{\hat{e}}_\theta$

Virtual work done in ~~moving~~ moving θ an angular displacement $\delta \theta$ by all non-conservative forces

$$\therefore \delta W = (\underline{F} + \underline{T}) \cdot \delta \underline{\theta} = \underline{F} \cdot \delta \underline{\theta}$$

$$\begin{aligned} &= \delta \theta (m a \cos \theta - T) \underline{\hat{e}}_r \cdot \underline{\hat{e}}_\theta - m a \delta \theta (m a \sin \theta) \underline{\hat{e}}_\theta \cdot \underline{\hat{e}}_\theta \\ &= -\delta \theta m a \sin \theta \end{aligned}$$

$$\therefore \delta W = 0 \quad \therefore \underline{a} = 0$$

∴ The Generalized force in coordinate θ is

$$Q_\theta = -m a l \sin \theta$$

Modified Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = Q_\theta$$

$$\therefore m l^2 \ddot{\theta} + m g l \sin \theta = -m a l \sin \theta$$

$$\therefore \ddot{\theta} + \frac{a+g}{l} \theta = 0$$

$$\therefore \omega = \sqrt{\frac{a+g}{l}} \quad \therefore \text{Period } P = 2\pi \sqrt{\frac{l}{a+g}}$$

$$\therefore \omega = \sqrt{\frac{a+g}{l}}$$

$$\therefore \text{period } P = 2\pi \sqrt{\frac{l}{a+g}}$$

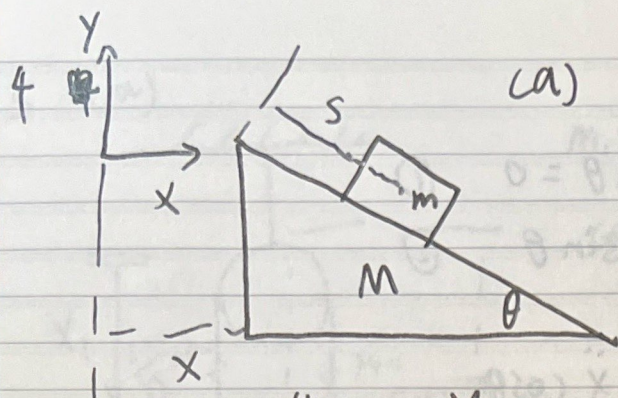
(c)

$$\text{When } a = -g, \quad P = \infty$$

$P \rightarrow \infty$ So when $a = -g$ the

pendulum does not oscillate

This is reasonable as when $a = -g$, the whole system is in free fall and there must be no oscillation.



(a)

$$\vec{V}_m \text{ rel. to } M = \vec{V}_{m/M}$$

let X, Y be the cartesian coordinate of m .

$$\text{then } X = x + s \cos \theta \quad Y = -s \sin \theta$$

$$T_m = \frac{1}{2} m (\dot{X}^2 + \dot{Y}^2) = \frac{1}{2} m [(\dot{x} + \dot{s} \cos \theta)^2 + (-\dot{s} \sin \theta)^2]$$

$$= \frac{1}{2} m [\dot{x}^2 + 2\dot{x}\dot{s} \cos \theta + \dot{s}^2 \cos^2 \theta + \dot{s}^2 \sin^2 \theta]$$

$$= \frac{1}{2} m [\dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s} \cos \theta]$$

$$T_m = \frac{1}{2} M \dot{x}^2$$

$$V_m = mgy = -mgs \sin \theta$$

$$V_M = 0$$

$$L = T - V = \frac{1}{2} m [\dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s} \cos \theta] + \frac{1}{2} M \dot{x}^2 + mgs \sin \theta$$

(b)

$$\textcircled{1} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \quad \therefore \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{d}{dt} (m\dot{x} + \dot{s} \cos \theta + M\dot{x})$$

$$= m\ddot{x} + m\ddot{s} \cos \theta + M\ddot{x}$$

$$\frac{\partial L}{\partial x} = 0$$

$$\therefore \textcircled{1} \quad (m+M)\ddot{x} + m\ddot{s} \cos \theta = 0$$

$$\textcircled{2} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{s}} \right) = \frac{\partial L}{\partial s} \quad \therefore \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{s}} \right) = \frac{d}{dt} (m\dot{s} + m\dot{x} \cos \theta)$$

$$\frac{\partial L}{\partial s} = mg \sin \theta$$

$$\therefore \textcircled{2} \quad m\ddot{s} + m\ddot{x} \cos \theta = mg \sin \theta$$

$$\therefore \begin{cases} (m+M)\ddot{x} + m\ddot{s}\cos\theta = 0 & (1) \\ \ddot{s} + \ddot{x}\cos\theta = g\sin\theta & (2) \end{cases}$$

$$(2) \Rightarrow \ddot{s} = g\sin\theta - \ddot{x}\cos\theta$$

substitute this into (1) we have :

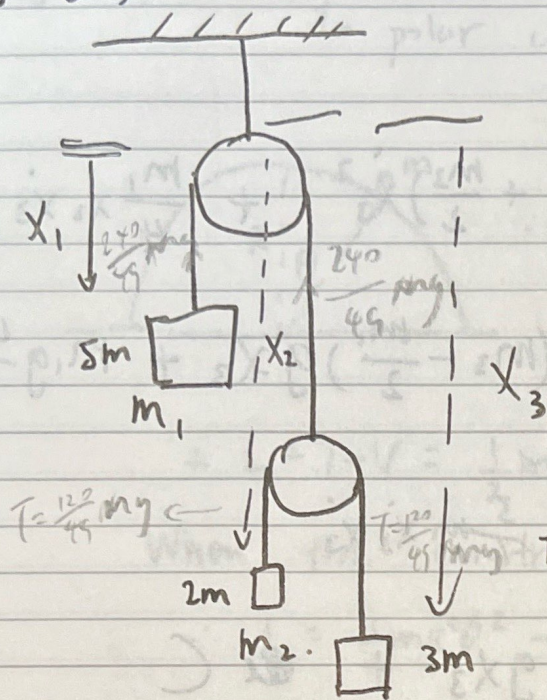
$$(m+M)\ddot{x} + m(g\sin\theta - \ddot{x}\cos\theta)\cos\theta = 0$$

$$\therefore (m+M)\ddot{x} - m\cos^2\theta\ddot{x} = -mg\sin\theta\cos\theta$$

$$\therefore \boxed{\ddot{x} = \frac{-mg\sin\theta\cos\theta}{(m+M) - m\cos^2\theta}} \quad \alpha \text{ is the}$$

acceleration of the ~~incl~~ incline

5 (a)



$$m_1 = 5m \quad m_2 = 2m \quad m_3 = 3m$$

constraint :

$$[x_2 - (l_1 - x_1)] + [x_3 - (l_2 - x_1)] = l_2$$

$$\therefore 2x_1 + x_2 + x_3 = 2l_1 + l_2$$

$$\text{let } L = 2l_1 + l_2$$

$$\text{then } 2x_1 + x_2 + x_3 = L$$

$$\therefore x_1 = \frac{L}{2} - \frac{x_2}{2} - \frac{x_3}{2}$$

$\therefore$ Motion can be described in two generalized coordinates.

(b)

$$T = \frac{1}{2} m_1 \dot{x}_1^2 \quad T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$= \frac{1}{2} m_1 \left(-\frac{\dot{x}_2}{2} - \frac{\dot{x}_3}{2} \right)^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$= \frac{1}{2} m_1 \left(\frac{1}{4} \dot{x}_2^2 + \frac{1}{4} \dot{x}_3^2 + \frac{1}{2} \dot{x}_2 \dot{x}_3 \right) + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$= \frac{1}{2} (5m) \left(\frac{1}{4} \dot{x}_2^2 + \frac{1}{4} \dot{x}_3^2 + \frac{1}{2} \dot{x}_2 \dot{x}_3 \right) + \frac{1}{2} (2m) \dot{x}_2^2 + \frac{1}{2} (3m) \dot{x}_3^2$$

$$= \left(\frac{5}{8} + 1 \right) m \dot{x}_2^2 + \left(\frac{5}{8} + \frac{3}{2} \right) m \dot{x}_3^2 + \frac{5}{4} m \dot{x}_2 \dot{x}_3$$

$$= \frac{13}{8} m \dot{x}_2^2 + \frac{11}{4} m \dot{x}_3^2 + \frac{5}{4} m \dot{x}_2 \dot{x}_3$$

$$= \left(\frac{m_1}{8} + \frac{m_2}{2} \right) \dot{x}_2^2 + \left(\frac{m_1}{8} + \frac{m_3}{2} \right) \dot{x}_3^2 + \frac{m_1}{4} \dot{x}_2 \dot{x}_3$$

$$V = -m_1 g x_1 - m_2 g x_2 - m_3 g x_3$$

$$= -m_1 g \left(\frac{L}{2} - \frac{x_2}{2} - \frac{x_3}{2} \right) - m_2 g x_2 - m_3 g x_3$$

$$= -m_1 g \frac{L}{2} - \left(m_2 - \frac{m_1}{2} \right) g x_2 - \left(m_3 - \frac{m_1}{2} \right) g x_3$$

$$\therefore L = T - V$$

$$= \left(\frac{m_1}{8} + \frac{m_2}{2}\right) \dot{x}_2^2 + \left(\frac{m_1}{8} + \frac{m_3}{2}\right) \dot{x}_3^2 + \frac{m_1}{4} \dot{x}_2 \dot{x}_3$$

$$+ \left(m_2 - \frac{m_1}{2}\right) g x_2 + \left(m_3 - \frac{m_1}{2}\right) g x_3 + m_1 g \frac{L}{2}$$

$$= \frac{13}{8} m \dot{x}_2^2 + \frac{17}{8} m \dot{x}_3^2 + \frac{5}{4} m \dot{x}_2 \dot{x}_3$$

$$- \frac{1}{2} m g x_2 + \frac{1}{2} m g x_3 + \text{const}$$

$$(b) \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) = \frac{\partial L}{\partial x_2}$$

$$\frac{13}{4} m \ddot{x}_2 + \frac{5}{4} m \dot{x}_3 = -\frac{1}{2} m g$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_3} \right) = \frac{\partial L}{\partial x_3}$$

$$\frac{17}{4} m \ddot{x}_3 + \frac{5}{4} m \dot{x}_2 = \frac{1}{2} m g$$

$$\text{Also } \ddot{x}_1 = -\frac{\dot{x}_2}{2} - \frac{\dot{x}_3}{2}$$

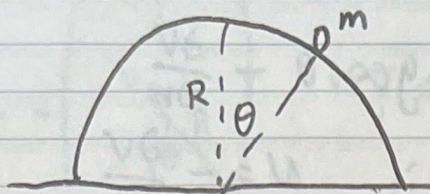
$$\therefore \begin{cases} \ddot{x}_1 = -\frac{1}{4g} g \\ \ddot{x}_2 = -\frac{11}{4g} g \\ \ddot{x}_3 = \frac{9}{4g} g \end{cases}$$

The pulley rotates because the tension is greater than $5mg$. The $5m$ block must accelerate upwards.

6 (a)

In polar coordinate

$$\underline{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}$$



$$\therefore T = \frac{1}{2} m \underline{v} \cdot \underline{v} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

$$V = mgy = mgr \cos \theta$$

$$\therefore L = T - V = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - mgr \cos \theta$$

When the constraint $r = R$ is valid, $\dot{r} = 0$

$$\therefore L = \frac{1}{2} m R^2 \dot{\theta}^2 - mgR \cos \theta$$

The only generalized coordinate is θ

$$\therefore \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} \quad \therefore \underline{m R^2 \ddot{\theta} = mgR \sin \theta}$$

$$\therefore \boxed{\ddot{\theta} = \frac{g}{R} \sin \theta}$$

(b) The Lagrangian now becomes

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - mgr \cos \theta - V(r)$$

We have $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta}$

$$\therefore \frac{d}{dt} (m r^2 \dot{\theta}) = mgr \sin \theta$$

$$\therefore m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = mgr \sin \theta$$

$$\therefore \cancel{2r\dot{r}\dot{\theta} + r^2\ddot{\theta}} = g \sin \theta \quad 2r\dot{\theta} + r\ddot{\theta} = g \sin \theta$$

$\therefore r$ varies infinitesimally $\therefore \ddot{r} \approx 0, \dot{r} \approx 0, r \approx R$

$$\therefore \cancel{r\ddot{\theta}} = \underline{\underline{\ddot{\theta} = \frac{g}{R} \sin \theta}} \quad (1)$$

We also have:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r}$$

$$\therefore m\ddot{r} = m r \dot{\theta}^2 - mg \cos \theta - \frac{\partial V}{\partial r}$$

The normal force N is $N = - \frac{\partial V}{\partial r}$.

and $\ddot{r} \approx 0$, $r \approx R$

$$\therefore \boxed{N = mg \cos \theta - m R \dot{\theta}^2} \quad (2)$$

(b)

$$(1) \Rightarrow \ddot{\theta} = \frac{g}{R} \sin \theta \quad \therefore \ddot{\theta} = \dot{\theta} \frac{d\dot{\theta}}{d\theta}$$

$$\therefore \dot{\theta} \frac{d\dot{\theta}}{d\theta} = \frac{g}{R} \sin \theta$$

$$\therefore \int \dot{\theta} d\dot{\theta} = \int \frac{g}{R} \sin \theta d\theta$$

$$\therefore \frac{1}{2} \dot{\theta}^2 = - \frac{g}{R} \cos \theta + C$$

Initial condition $\dot{\theta} = 0$ when $\theta = 0$

$$\therefore C = \frac{g}{R}$$

$$\therefore \frac{1}{2} \dot{\theta}^2 = \frac{g}{R} (1 - \cos \theta) \quad \therefore \dot{\theta}^2 = \frac{2g}{R} (1 - \cos \theta)$$

Substitute into (2) $\Rightarrow N = mg \cos \theta - m R \dot{\theta}^2 = mg \cos \theta - m R \left(\frac{2g}{R} \right) (1 - \cos \theta)$

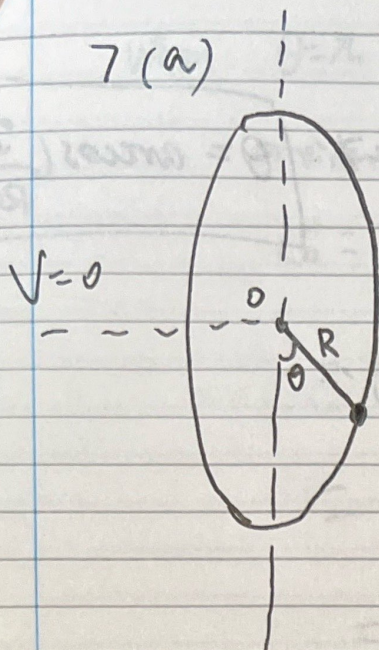
$$\therefore N = mg \cos \theta - 2mg + 2mg \cos \theta$$

$$\therefore N = 3mg \cos \theta - 2mg$$

When particle leaves the surface $N = 0$, $\theta = \theta_{\max}$

$$\therefore 3mg \cos \theta_{\max} - 2mg = 0$$

$$\therefore \cancel{\theta_{\max}} \quad \therefore \boxed{\cos \theta_{\max} = \frac{2}{3}}$$



In spherical coordinates:

$$\underline{v} = \dot{r} \underline{\hat{e}}_r + r \dot{\phi} \sin \theta \underline{\hat{e}}_\phi + r \dot{\theta} \underline{\hat{e}}_\theta$$

$$T = \frac{1}{2} m \underline{v} \cdot \underline{v}$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 \sin^2 \theta + r^2 \dot{\theta}^2)$$

Condition of constraints: $\dot{r} = 0$ and $\dot{\phi} = \omega = \text{const.}$

$$\therefore T = \frac{1}{2} m (R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \omega^2)$$

$$V = -m g R \cos \theta$$

$$\therefore L = T - V = \frac{1}{2} m (R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \omega^2) + m g R \cos \theta$$

~~Total energy $E = T + V = \frac{1}{2} m (R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \omega^2) - m g R \cos \theta$~~

~~$= \frac{1}{2} m R^2 \dot{\theta}^2 + \underbrace{\left(\frac{1}{2} m R^2 \sin^2 \theta \omega^2 - m g R \cos \theta \right)}_{= V_{\text{eff}}(\theta)}$~~

~~$V_{\text{eff}}(\theta)$ is the effective potential~~

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta}$$

$$\therefore m R^2 \ddot{\theta} = m R^2 \omega^2 \sin \theta \cos \theta - m g R \sin \theta$$

$$R \ddot{\theta} = \sin \theta (R \omega^2 \cos \theta - g)$$

~~In some cases~~ In equilibrium positions $\ddot{\theta} = 0$

$$\therefore m R^2 \omega^2 \sin \theta \cos \theta - m g R \sin \theta = 0$$

$$\therefore \sin \theta (R \omega^2 \cos \theta - g) = 0$$

$$(0 \leq \theta \leq \pi)$$

$$\sin \theta = 0 \Rightarrow \boxed{\theta = 0 \text{ or } \theta = \pi}$$

$$R\omega^2 \cos \theta - g = 0 \Rightarrow \cos \theta = \frac{g}{R\omega^2} \Rightarrow \boxed{\theta = \arccos\left(\frac{g}{R\omega^2}\right)}$$

(a) $\therefore$ When ~~$R\omega^2 < g$~~ $\omega^2 < \frac{g}{R}$,

The equilibrium points are

$$\theta = 0 \text{ and } \theta = \pi.$$

When $\omega^2 > \frac{g}{R}$

The equilibrium points are

$$\theta = 0, \theta = \pi, \text{ and } \theta = \arccos\left(\frac{g}{R\omega^2}\right)$$

(b)

When $\theta = 0$

We consider small variation δ of angle and set

$\theta = 0 + \delta$ when δ is very small

$$\ddot{\theta} = \ddot{\delta} \quad \sin \theta (R\omega^2 \cos \theta - g) = \delta (R\omega^2 - g)$$

$$\therefore R\ddot{\delta} = (R\omega^2 - g)\delta \quad \underline{\ddot{\delta} = \left(\omega^2 - \frac{g}{R}\right)\delta}$$

$\therefore$ The equilibrium is stable if $\omega^2 < \frac{g}{R}$

and is unstable if $\omega^2 \geq \frac{g}{R}$

When $\theta = \pi$

consider ~~$\theta = \pi + \delta$~~ $\theta = \pi + \delta$

$$R\ddot{\delta} = \sin(\pi + \delta) (R^2 \omega^2 \cos(\pi + \delta) - g)$$

$$= \left(\underbrace{\sin \pi}_{0} \underbrace{\cos \delta}_{1} + \underbrace{\cos \pi}_{-1} \underbrace{\sin \delta}_{\delta} \right) \left(R^2 \omega^2 \left(\underbrace{\cos \pi}_{-1} \underbrace{\cos \delta}_{1} - \underbrace{\sin \pi}_{0} \underbrace{\sin \delta}_{\delta} \right) - g \right)$$

$$= -\sin \delta (-R\omega^2 - g)$$

$$= \delta (R\omega^2 + g)$$

$$\therefore \ddot{\delta} = \delta \left(\omega^2 + \frac{g}{R} \right)$$

$\therefore$ The equilibrium is unstable

When $\theta = \arccos\left(\frac{g}{R\omega^2}\right)$ ~~$\theta = 0$~~ $\cos \theta = \frac{g}{R\omega^2}$

$$\text{let } \theta = \arccos\left(\frac{g}{R\omega^2}\right) + \delta$$

$$R\ddot{\delta} = \sin\left(\cos^{-1}\left(\frac{g}{R\omega^2}\right) + \delta\right) \left[R\omega^2 \cos\left(\cos^{-1}\left(\frac{g}{R\omega^2}\right) + \delta\right) - g \right]$$

$$\text{let } \phi = \arccos\left(\frac{g}{R\omega^2}\right) \quad \cos \phi = \frac{g}{R\omega^2} \quad \sin \phi = \sqrt{1 - \frac{g^2}{R^2 \omega^4}}$$

$$R\ddot{\delta} = \sin(\phi + \delta) [R\omega^2 \cos(\phi + \delta) - g]$$

$$\therefore R\ddot{\delta} = (\sin \phi \cos \delta + \cos \phi \sin \delta) [R\omega^2 (\cos \phi \cos \delta - \sin \phi \sin \delta) - g]$$

$$\Rightarrow R\ddot{\delta} = (\sin \phi + \cos \phi \delta) [R\omega^2 (\cos \phi - \sin \phi \delta) - g]$$

$$= (\sin \phi + \cos \phi \delta) [-R\omega^2 \sin \phi \delta]$$

$$\because \delta^2 \text{ is very small} \quad \therefore R\ddot{\delta} = -R\omega^2 \sin^2 \phi \delta$$

$$\therefore \ddot{\delta} = -\omega^2 \sin^2 \phi \delta$$

∴ The equilibrium is stable,
 Frequencies of oscillations:

For $\theta = 0$, $\omega^2 < \frac{g}{R}$

$$\ddot{\delta} + \left(\frac{g}{R} - \omega^2 \right) \delta = 0$$

$$\ddot{\delta} + \left(\frac{g}{R} - \omega^2 \right) \delta = 0$$

$$\omega_0 = \sqrt{\frac{g}{R} - \omega^2}$$

For $\theta = \arccos\left(\frac{g}{R\omega^2}\right)$, $\omega^2 > \frac{g}{R}$

$$\ddot{\delta} + \omega^2 \sin^2 \phi \delta = 0$$

$$\sin \phi = \sqrt{1 - \frac{g^2}{R^2 \omega^4}} \Rightarrow \sin^2 \phi = 1 - \frac{g^2}{R^2 \omega^4}$$

$$\omega_1 = \omega \sin \phi = \omega \sqrt{1 - \frac{g^2}{R^2 \omega^4}}$$

$$= \omega \sqrt{1 - \frac{g^2}{R^2 \omega^4}}$$

$$\omega_1 = \sqrt{\omega^2 - \frac{g^2}{R^2 \omega^2}}$$

(C) The Hamiltonian H is $H = \sum_i p_i \dot{q}_i - L$

~~How~~ $\therefore L = L(\theta, \dot{\theta}, t)$

$$\therefore H = P_\theta \dot{\theta} - L$$

$$L = \frac{1}{2} m (R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \omega^2) + mgR \cos \theta$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2 \dot{\theta} \quad \therefore \dot{\theta} = \frac{P_\theta}{mR^2}$$

$$\therefore H = mR^2 \dot{\theta}^2 - L = \frac{1}{2} m (R^2 \dot{\theta}^2 - R^2 \sin^2 \theta \omega^2) - mgR \cos \theta$$

$$\therefore H = \frac{1}{2} m \left(\frac{P_\theta^2}{m^2 R^2} - R^2 \sin^2 \theta \omega^2 \right) - mgR \cos \theta$$

* Total Energy $T+V = \frac{1}{2} m (R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \omega^2) - mgR \cos \theta$

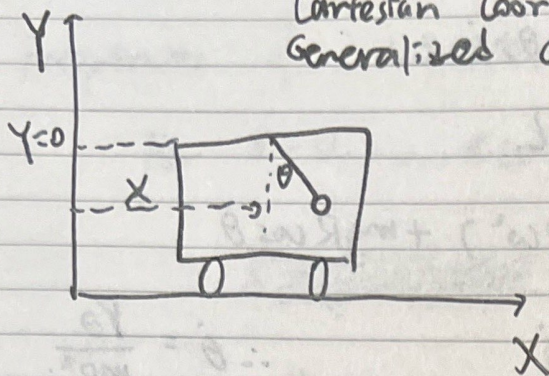
The Hamiltonian H is a constant (conserved) because the Lagrangian L is not time dependent.

* The total ~~the~~ Energy $T+V$ is not conserved as the angular ω speed of hoop ω is constant.

This must be achieved by ^{supplying} ~~putting~~ ^{to to or from} or removing energy ~~the~~ the system. The system is not isolated.

8.)

The mass has
Cartesian coordinates X, Y
Generalized coordinates X, θ



$$X = X + l \sin \theta$$

$$Y = -l \cos \theta$$

$$T_m = \frac{1}{2} m (\dot{X}^2 + \dot{Y}^2) \quad \text{is the kinetic energy of mass}$$

~~$$\dot{X} = \dot{X} + l \cos \theta \dot{\theta}$$~~

$$\dot{X} = \dot{X} + l \cos \theta \dot{\theta} \quad \dot{Y} = l \sin \theta \dot{\theta}$$

$$T_m = \frac{1}{2} m (\dot{X}^2 + \dot{Y}^2) = \frac{1}{2} m [(\dot{X} + l \cos \theta \dot{\theta})^2 + (l \sin \theta \dot{\theta})^2]$$

$$= \frac{1}{2} m [\dot{X}^2 + 2\dot{X}\dot{\theta}l \cos \theta + l^2 \cos^2 \theta \dot{\theta}^2 + l^2 \sin^2 \theta \dot{\theta}^2]$$

$$= \frac{1}{2} m [\dot{X}^2 + 2\dot{X}\dot{\theta}l \cos \theta + l^2 \dot{\theta}^2]$$

Kinetic energy of the box $T_M = \frac{1}{2} M \dot{X}^2$

$\therefore$ Total kinetic energy

$$T = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m [\dot{X}^2 + l^2 \dot{\theta}^2 + 2\dot{X}\dot{\theta}l \cos \theta]$$

Total potential energy $V = mgy = -mgl \cos \theta$

$\therefore$ The Lagrangian

$$L = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m [\dot{X}^2 + l^2 \dot{\theta}^2 + 2\dot{X}\dot{\theta}l \cos \theta] + mgl \cos \theta$$

The equations of motion:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \quad \frac{\partial L}{\partial \dot{x}} = M\dot{x} + m\dot{x} + ml \cos \theta \dot{\theta}$$

$$\frac{\partial L}{\partial x} = 0$$

$$\therefore \frac{d}{dt} (M\dot{x} + m\dot{x} + ml \cos \theta \dot{\theta}) = 0 \quad (1)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta}$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2 \ddot{\theta} + ml \dot{x} \cos \theta$$

$$\frac{\partial L}{\partial \theta} = -ml \dot{x} \dot{\theta} \sin \theta - mgl \sin \theta$$

$$\therefore \frac{d}{dt} (ml^2 \ddot{\theta} + ml \dot{x} \cos \theta) = -ml \dot{x} \dot{\theta} \sin \theta - mgl \sin \theta \quad (2)$$

$$(1) \Rightarrow M\ddot{x} + m\ddot{x} + ml(-\sin \theta \dot{\theta}^2 + \cos \theta \ddot{\theta}) = 0$$

$$\therefore \cancel{M\ddot{x} + m\ddot{x}} \quad \therefore (M+m)\ddot{x} = \cancel{-ml \sin \theta \dot{\theta}^2}$$

$$\therefore \boxed{M\ddot{x} + m\ddot{x} - ml \sin \theta \dot{\theta}^2 + ml \cos \theta \ddot{\theta} = 0}$$

$$(2) \Rightarrow ml^2 \ddot{\theta} + ml(-\dot{x} \sin \theta \dot{\theta} + \ddot{x} \cos \theta)$$

$$= -ml \dot{x} \dot{\theta} \sin \theta - mgl \sin \theta$$

$$\therefore ml^2 \ddot{\theta} + ml \ddot{x} \cos \theta + mgl \sin \theta = 0$$

$$\therefore \cancel{ml^2 \ddot{\theta} + ml \ddot{x} \cos \theta + mgl \sin \theta}$$

$$\therefore \boxed{\ddot{\theta} + \frac{\ddot{x}}{l} \cos \theta + \frac{g}{l} \sin \theta = 0}$$

From (1) we have.

$$(M+m)\ddot{x} = ml \sin \theta \dot{\theta}^2 - ml \cos \theta \ddot{\theta}$$

$$\therefore \ddot{x} = \frac{ml \sin \theta \dot{\theta}^2 - ml \cos \theta \ddot{\theta}}{M+m}$$

$$\therefore \frac{m \sin \theta \cos \theta \dot{\theta}^2 - m \cos^2 \theta \ddot{\theta}}{M+m} + \ddot{\theta} + \frac{g}{l} \sin \theta = 0 \quad (3)$$

~~$$\ddot{\theta} = \frac{g}{l} \sin \theta$$~~

$\therefore \frac{\partial L}{\partial x} = 0 \quad \therefore$ The conjugate momentum of coordinate x is ~~conserved~~, which is the linear momentum of the whole system, is conserved

(b) For small angles, θ and $\dot{\theta}$ are small, $\sin \theta \approx \theta$, $\cos \theta \approx 1$, $\theta^2 \approx \dot{\theta}^2 \approx 0$

$\therefore$ (3) becomes

$$\left(\frac{M}{M+m} \right) \ddot{\theta} + \frac{g}{l} \theta = 0$$

$$\therefore \ddot{\theta} + \frac{g}{l} \left(\frac{M+m}{M} \right) \theta = 0$$

$$\text{let } \omega = \left[\frac{M+m}{M} \right]^{1/2} \left(\frac{g}{l} \right)^{1/2}, \text{ then } \theta = \theta_0 \cos(\omega t + \phi)$$

① becomes $\ddot{x} = - \frac{ml}{M+m} \ddot{\theta}$

Integrate twice gives $x = - \frac{ml}{M+m} \theta + C_1 t + D$

$$\therefore x = - \frac{ml \theta_0}{M+m} \cos(\omega t + \phi) + C_1 t + D$$

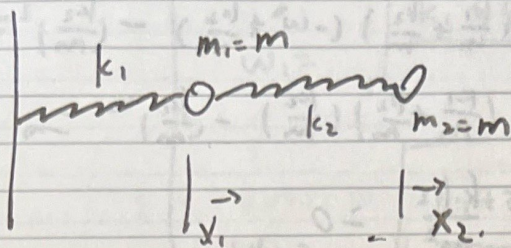
$C = \phi = 0$
 ~~$C = \phi = 0$~~
When initial velocity of the box and the mass are both 0

θ (At $t=0$, $\theta = \theta_0$) $\therefore$ ~~$\theta = \theta_0$~~

$\therefore$ pendulum and mass ~~undergo~~ ^{undergo} SHM about the centre of mass of the system with ~~angular~~ angular frequency

$$\omega = \sqrt{\frac{M+m}{M} \frac{g}{l}}$$

9(a) $\therefore$ equal mass $\therefore m_1 = m_2 = m$



$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2)$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$\therefore L = T - V = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) = \frac{\partial L}{\partial x_1} \quad \frac{\partial L}{\partial \dot{x}_1} = m \dot{x}_1 \quad \frac{\partial L}{\partial x_1} = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$\therefore m \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) = \frac{\partial L}{\partial x_2} \quad \frac{\partial L}{\partial \dot{x}_2} = m \dot{x}_2 \quad \frac{\partial L}{\partial x_2} = -k_2 (x_2 - x_1)$$

$$\therefore m \ddot{x}_2 = -k_2 (x_2 - x_1)$$

(b)

$$\therefore \begin{pmatrix} \frac{d^2}{dt^2} + \frac{k_1}{m} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{d^2}{dt^2} + \frac{k_2}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

try $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \text{Re} \left(\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \right) e^{i\omega t}$ $\frac{d^2}{dt^2} = -\omega^2$

$$\therefore \begin{pmatrix} -\omega^2 + \frac{k_1}{m} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & -\omega^2 + \frac{k_2}{m} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A

For non-trivial solutions to exist.

$$\det(A) = 0 \quad \therefore (-\omega^2 + \frac{k_1}{m} + \frac{k_2}{m}) (-\omega^2 + \frac{k_2}{m}) - (\frac{k_2}{m})^2 = 0$$

$$\therefore \omega^4 - (\frac{k_1}{m} + \frac{2k_2}{m}) \omega^2 + (\frac{k_1}{m} + \frac{k_2}{m}) (\frac{k_2}{m}) - (\frac{k_2}{m})^2 = 0$$

$$\therefore \omega^4 - (\frac{k_1}{m} + \frac{2k_2}{m}) \omega^2 + \frac{k_1 k_2}{m^2} = 0$$

$$\therefore \omega^2 = \frac{\frac{k_1}{m} + \frac{2k_2}{m} \pm \sqrt{(\frac{k_1}{m} + \frac{2k_2}{m})^2 - \frac{4k_1 k_2}{m^2}}}{2}$$

$$= \frac{\frac{k_1}{m} + \frac{2k_2}{m} \pm \sqrt{\frac{k_1^2}{m^2} + \frac{4k_1^2}{m^2} + \frac{4k_1 k_2}{m^2} - \frac{4k_1 k_2}{m^2}}}{2}$$

$$= \frac{\frac{k_1}{m} + \frac{2k_2}{m} \pm \sqrt{\frac{k_1^2}{m^2} + \frac{4k_2^2}{m^2}}}{2}$$

$$\therefore \omega_1 = \sqrt{\frac{\frac{k_1}{m} + \frac{2k_2}{m} + \sqrt{\frac{k_1^2}{m^2} + \frac{4k_2^2}{m^2}}}{2}}$$

$$\omega_2 = \sqrt{\frac{\frac{k_1}{m} + \frac{2k_2}{m} - \sqrt{\frac{k_1^2}{m^2} + \frac{4k_2^2}{m^2}}}{2}}$$

In normal mode 1: $\omega = \omega_1$

$$\begin{pmatrix} \omega_1^2 - \frac{k_1}{m} - \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \omega_1^2 - \frac{k_2}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \frac{x_1}{x_2} = \frac{-\frac{k_2}{m}}{\omega_1^2 - \frac{k_2}{m} - \frac{k_1}{m}}$$

∴ two normal modes.

$$\omega_1 = \sqrt{\frac{k_1 + 2k_2 + \sqrt{k_1^2 + 4k_2^2}}{2m}}$$

$$\omega_2 = \sqrt{\frac{k_1 + 2k_2 - \sqrt{k_1^2 + 4k_2^2}}{2m}}$$

In normal mode 1: $\omega = \omega_1$,

$$\begin{pmatrix} \frac{k_1 - \sqrt{k_1^2 + 4k_2^2}}{2m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & -\frac{k_1 - \sqrt{k_1^2 + 4k_2^2}}{2m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \frac{x_1}{x_2} = -\frac{k_1 + \sqrt{k_1^2 + 4k_2^2}}{2k_2}$$

∴ Solution is

$$x_1 = -\frac{k_1 + \sqrt{k_1^2 + 4k_2^2}}{2k_2} A \cos(\omega_1 t + \phi_1)$$

$$x_2 = A \cos(\omega_1 t + \phi_1)$$

In normal mode 2: $\omega = \omega_2$

$$\begin{pmatrix} \frac{\sqrt{k_1^2 + 4k_2^2} + k_1}{2m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{\sqrt{k_1^2 + 4k_2^2} - k_1}{2m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \frac{x_1}{x_2} = \frac{\sqrt{k_1^2 + 4k_2^2} - k_1}{2k_2}$$

$\therefore$ Solution is

$$\boxed{\begin{aligned} x_1 &= \frac{\sqrt{k_1^2 + 4k_2^2} - k_1}{2k_2} B \cos(\omega_2 t + \phi_2) \\ x_2 &= B \cos(\omega_2 t + \phi_2) \end{aligned}}$$

Special cases:

when $k_1 \ll k_2$

$$\text{mode 1: } \frac{x_1}{x_2} = -1$$

This is the motion relative to the centre of mass

$$\text{mode 2: } \frac{x_1}{x_2} = 1$$

This is the centre of mass motion.

when $k_1 \gg k_2$

$$\text{mode 1: } \frac{x_1}{x_2} = -\frac{k_1}{k_2} \rightarrow -\infty$$

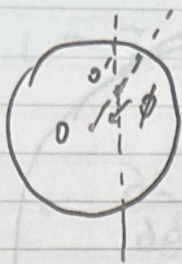
($\frac{x_2}{x_1} \rightarrow 0$)

This corresponds to m_1 moves with SHM but m_2 remains stationary.

$$\text{mode 2: } \frac{x_1}{x_2} = 0 \quad \left(\frac{x_2}{x_1} \rightarrow \infty\right)$$

This correspond to m_1 remain stationary but m_2 ~~move~~ moves with SHM.

10.



$$I = I_c + mx^2 = \frac{1}{2}ma^2 + mx^2$$

$$T = \frac{1}{2}I\dot{\phi}^2 \quad V = -mgx \cos \phi$$

$$\therefore L = T - V = \frac{1}{2}I\dot{\phi}^2 + mgx \cos \phi$$

~~$$\therefore L = \frac{1}{2} \left(\frac{1}{2}ma^2 + mx^2 \right) \dot{\phi}^2 + mgx \cos \phi$$~~

$$\frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi}$$

$$\frac{\partial L}{\partial \phi} = -mgx \sin \phi$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \quad \therefore I\ddot{\phi} = -mgx \sin \phi$$

$\therefore \phi$ is small $\therefore \sin \phi \approx \phi$

$$\therefore \ddot{\phi} + \frac{mgx}{I} \phi = 0$$

$\therefore \omega$ is Angular frequency $\omega = \sqrt{\frac{mgx}{I}}$

~~$$\therefore \omega = \sqrt{\frac{mgx}{I}}$$~~

$$\therefore T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{mgx}}$$

$$\therefore T = 2\pi \sqrt{\frac{\frac{1}{2}ma^2 + mx^2}{mgx}} = 2\pi \sqrt{\frac{\frac{a^2}{2gx} + \frac{x}{g}}{1}}$$

T is minimized when $\frac{a^2}{2gx} + \frac{x}{g}$ is

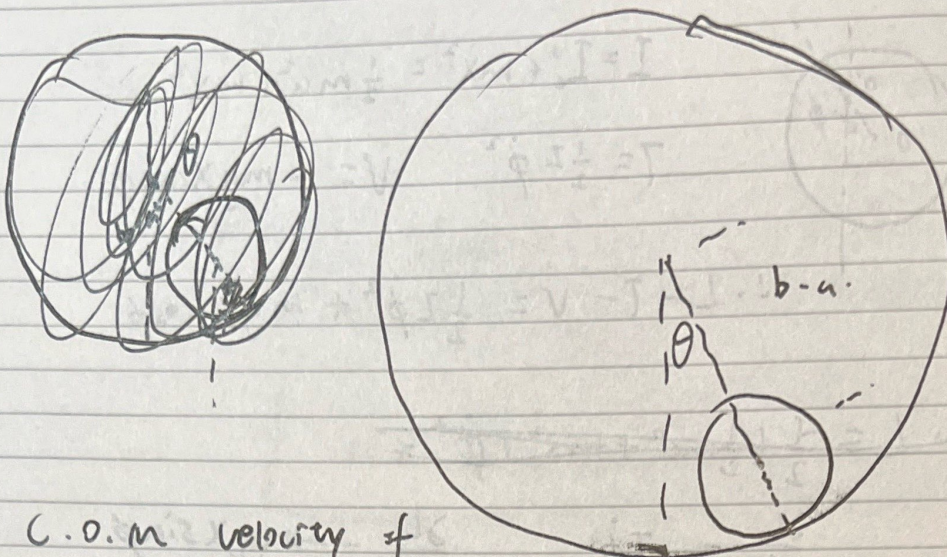
minimized.

For

$$f'(x) = -\frac{a^2}{2gx^2} + \frac{1}{g} = 0$$

$$\therefore x^2 = \frac{a^2}{2} \quad \therefore x = \pm \frac{a}{\sqrt{2}}$$

11



C.O.M. velocity of small cylinder is $V = \omega a$ where ω is the ~~angular~~ angular velocity, with respect to its C.O.M.

Also $V = (b-a)\dot{\theta}$

$$\therefore (b-a)\dot{\theta} = \omega a \quad \therefore \omega = \frac{b-a}{a} \dot{\theta}$$

$$T_r = \frac{1}{2} I \omega^2 = \frac{1}{2} \left(\frac{1}{2} m a^2 \right) \omega^2 = \frac{1}{4} m a^2 \frac{(b-a)^2}{a^2} \dot{\theta}^2$$

$$= \frac{1}{4} m (b-a)^2 \dot{\theta}^2 \quad \text{is the rotational kinetic energy.}$$

~~$V = (b-a)\dot{\theta}$~~ $V = \frac{b-a}{a} \dot{\theta}$

$$T_t = \frac{1}{2} m V^2 = \frac{1}{2} m (b-a)^2 \dot{\theta}^2 \quad \text{is the translational kinetic energy}$$

~~Total~~ Total kinetic energy

$$T = T_t + T_r = \frac{3}{4} m (b-a)^2 \dot{\theta}^2$$

Potential Energy

$$V = -mg(b-a) \cos \theta$$

$$L = T - V = \frac{3}{4} m (b-a)^2 \dot{\theta}^2 + mg(b-a) \cos \theta$$

$$\therefore \frac{\partial L}{\partial \dot{\theta}} = \frac{3}{2} m (b-a)^2 \dot{\theta} \quad \frac{\partial L}{\partial \theta} = -mg(b-a) \sin \theta$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} = 0$$

$$\therefore \frac{3}{2} m (b-a)^2 \ddot{\theta} + mg(b-a) \sin \theta = 0$$

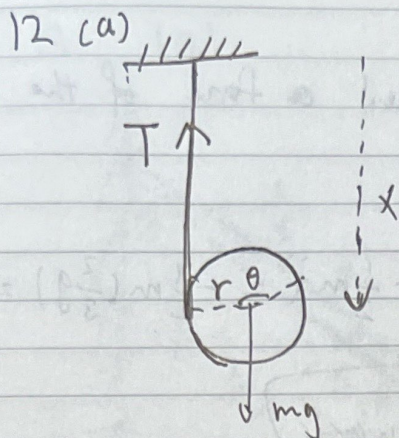
for small θ $\sin \theta \approx \theta$

$$\therefore \frac{3}{2} (b-a) \ddot{\theta} + g \theta = 0$$

$$\therefore \ddot{\theta} + \frac{g}{\left[\frac{3}{2} (b-a) \right]} \theta = 0$$

$$\therefore \text{period } T = 2\pi \sqrt{\frac{\frac{3}{2} (b-a)}{2g}}$$

which is equivalent to a pendulum of ~~period~~ length $\frac{3(b-a)}{2}$.



$$I = \frac{1}{2}mr^2$$

Potential Energy $V = -mgx$

Kinetic Energy $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$

Constraint : ~~$f(x, \theta) = x - r\theta = 0$~~
 $f(x, \theta) = x - r\theta = 0$

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{4}mr^2\dot{\theta}^2 + mgx$$

$$f = x - r\theta$$

$$\therefore \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \lambda(t) \frac{\partial f}{\partial x} = 0 \quad (1)$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + \lambda(t) \frac{\partial f}{\partial \theta} = 0 \quad (2)$$

$$(1) \Rightarrow mg - m\ddot{x} + \lambda = 0$$

$$(2) \Rightarrow -\frac{1}{2}mr^2\ddot{\theta} - \lambda r = 0$$

~~$\ddot{x} = \ddot{\theta}$~~ $\because f(x, \theta) = 0 \therefore x - r\theta = 0$

$$\therefore \ddot{x} - r\ddot{\theta} = 0 \therefore \ddot{\theta} = \frac{\ddot{x}}{r}$$

$$(1) \therefore mg - m\ddot{x} + \lambda = 0, \quad -\frac{1}{2}mr\ddot{x} - \lambda r = 0$$

~~$\lambda = -\frac{1}{2}m\ddot{x}$~~ $\therefore \lambda = -\frac{1}{2}m\ddot{x}$

substitute into (1) $\Rightarrow mg - \frac{3}{2}m\ddot{x} = 0$

$$\therefore \ddot{x} = \frac{2}{3}g$$

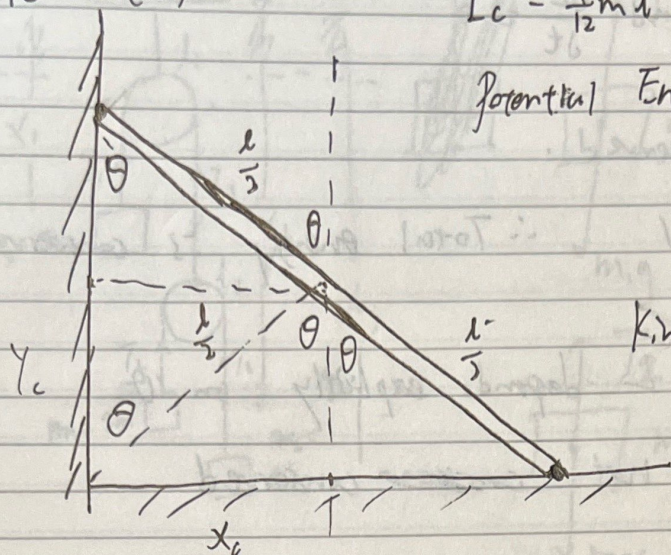
$\therefore$ Initial acceleration is $\boxed{\frac{2}{3}g}$

The tension T is the generalized force of the coordinate x

$$\therefore T = Q_x = \lambda \frac{\partial f}{\partial x} = -\frac{1}{2} m \ddot{x} = -\frac{1}{2} m \left(\frac{2}{3} g \right) = -\frac{mg}{3}$$

$$\therefore \text{Tension is } \boxed{\frac{mg}{3} \text{ upwards}}$$

13 (a)



$$I_c = \frac{1}{12} m l^2 \quad \cancel{V_{cm} = \frac{1}{2} m g \frac{l}{2} \cos \theta} \quad V_{cm} = \frac{1}{2} m g \frac{l}{2} \cos \theta$$

$$\text{Potential Energy: } V = m g \left(\frac{l}{2} \cos \theta \right)$$

$$= \frac{1}{2} m g l \cos \theta$$

$$\text{Kinetic Energy: } T = T_{cm} + T_{rel}$$

$$T_{cm} = \frac{1}{2} m V_{cm}^2 = \frac{1}{2} m \left(\frac{l}{2} \dot{\theta} \right)^2$$

$$= \frac{1}{8} m l^2 \dot{\theta}^2$$

$$T_{rel} = \frac{1}{2} I_c \dot{\theta}^2 = \frac{1}{2} \left(\frac{1}{12} m l^2 \right) \dot{\theta}^2 = \frac{1}{24} m l^2 \dot{\theta}^2$$

$$\therefore T = \left(\frac{1}{8} + \frac{1}{24} \right) m l^2 \dot{\theta}^2 = \frac{1}{6} m l^2 \dot{\theta}^2$$

(b)

$$T = \frac{1}{2} m (\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{24} m l^2 \dot{\theta}^2$$

$$\text{Constraints: } x_c = \frac{l}{2} \sin \theta \quad y_c = \frac{l}{2} \cos \theta$$

$$\therefore \dot{x}_c = \frac{l}{2} \cos \theta \dot{\theta} \quad \dot{y}_c = -\frac{l}{2} \sin \theta \dot{\theta}$$

$$\therefore \dot{x}_c^2 + \dot{y}_c^2 = \frac{l^2}{4} (\cos^2 \theta + \sin^2 \theta) = \frac{l^2}{4}$$

$$T = \frac{1}{2} m \left(\frac{l^2}{4} \right) + \frac{1}{24} m l^2 \dot{\theta}^2 = \frac{1}{6} m l^2 \dot{\theta}^2$$

$$\therefore L = \frac{1}{6} m l^2 \dot{\theta}^2 - \frac{1}{2} m g l \cos \theta$$

(c)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{3} m l^2 \dot{\theta}$$

$$\text{Hamiltonian } H = p_\theta \dot{\theta} - L = \cancel{\frac{1}{2} m l^2 \dot{\theta}^2} - \frac{1}{6} m l^2 \dot{\theta}^2 + \frac{1}{2} m g l \cos \theta$$

$$= \frac{1}{3} m l^2 \dot{\theta}^2 - \frac{1}{6} m l^2 \dot{\theta}^2 + \frac{1}{2} m g l \cos \theta$$

$$= \frac{1}{6} m l^2 \dot{\theta}^2 + \frac{1}{2} m g l \cos \theta = T + V$$

$$\therefore \frac{dH}{dt} = -\frac{\partial L}{\partial t} \quad \text{and} \quad \frac{\partial L}{\partial t} = 0$$

$\therefore H$ is conserved.

$\therefore H = T + V$ $\therefore$ Total energy is conserved

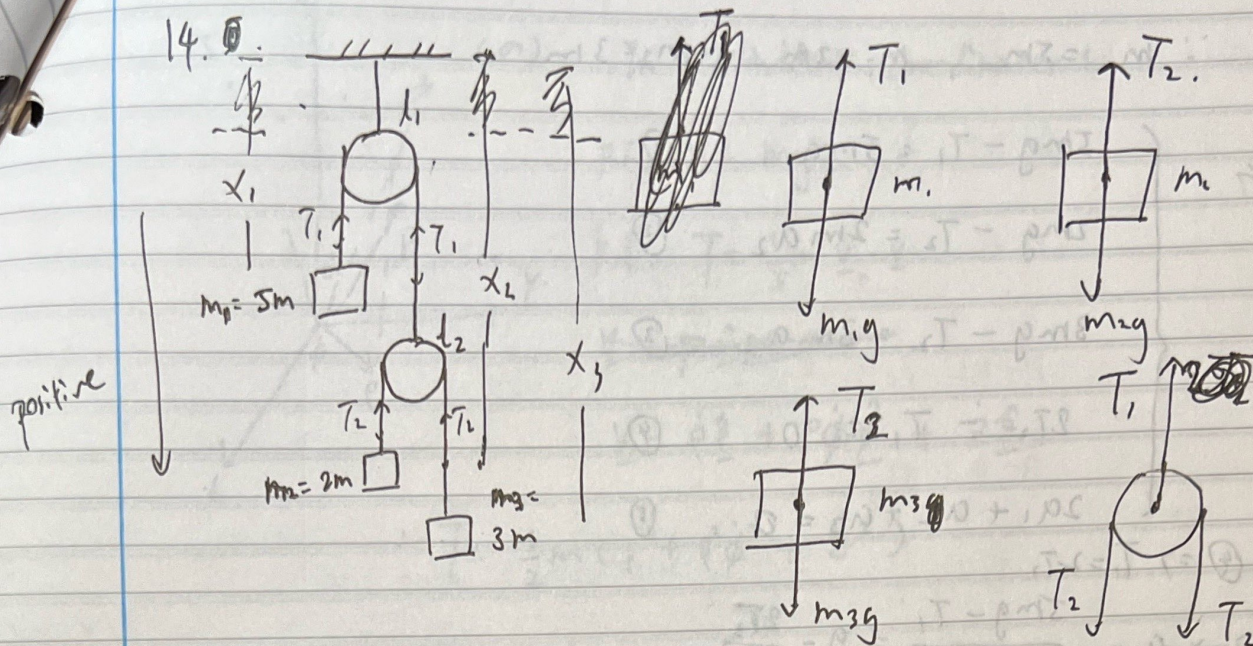
$\therefore$ Lagrangian L depends explicitly on θ

$\therefore T$ is not ~~conserved~~ conserved

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} \quad \therefore \frac{1}{3} m l^2 \ddot{\theta} = \frac{1}{2} m g l \sin \theta$$

$$\therefore \frac{2l}{3} \ddot{\theta} = g \sin \theta$$

$$\therefore \boxed{\ddot{\theta} - \frac{3g}{2l} \sin \theta = 0} \quad \text{is the E.O.M}$$



Free

$$\therefore m_1 g - T_1 = m_1 a_1 \quad (1)$$

$$m_2 g - T_2 = m_2 a_2 \quad (2)$$

$$m_3 g - T_2 = m_3 a_3 \quad (3)$$

$$2T_2 - T_1 = 0 \quad (4)$$

The (4) equation comes from the fact that light pulley cannot experience unbalanced forces, otherwise it will have infinite acceleration.

$$\therefore \text{for } l_1 = x_1 + (x_2 - (l_1 - x_1))$$

$$x_2 - (l_1 - x_1) + x_3 - (l_1 - x_1) = l_2$$

$$\therefore 2x_1 + x_2 + x_3 = 2l_1 + l_2$$

$$\therefore 2\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 = 0$$

$$\therefore 2a_1 + a_2 + a_3 = 0 \quad (5)$$

Equations (1), (2), (3), (4), (5) are used to solve for T_1, T_2, a_1, a_2, a_3 .

$$\therefore m_1 = 5m \quad m_2 = 2m \quad m_3 = 3m$$

$$\begin{cases} 5mg - T_1 = 5ma_1 & (1) \\ 2mg - T_2 = 2ma_2 & (2) \\ 3mg - T_2 = 3ma_3 & (3) \\ 2T_2 - T_1 = 0 & (4) \\ 2a_1 + a_2 + a_3 = 0 & (5) \end{cases}$$

$$(4) \Rightarrow T_1 = 2T_2$$

$$(1) \Rightarrow a_1 = \frac{5mg - T_1}{5m} = g - \frac{2T_2}{5m}$$

$$(2) \Rightarrow a_2 = g - \frac{T_2}{2m} \quad (3) \Rightarrow a_3 = g - \frac{T_2}{3m}$$

$$\text{Sub into (5)} \Rightarrow 2\left(g - \frac{2T_2}{5m}\right) + \left(g - \frac{T_2}{2m}\right) + \left(g - \frac{T_2}{3m}\right) = 0$$

$$\therefore 4g - \frac{49T_2}{30m} = 0$$

$$\therefore T_2 = \frac{120}{49} mg$$

$$\therefore T_1 = \frac{240}{49} mg$$

$$\text{Sub into (1), (2), (3)} \Rightarrow$$

$$a_1 = \frac{1}{49} g$$

$$a_2 = -\frac{11}{49} g$$

$$a_3 = \frac{9}{49} g$$

The Lagrangian method

ignores the forces of constraints,

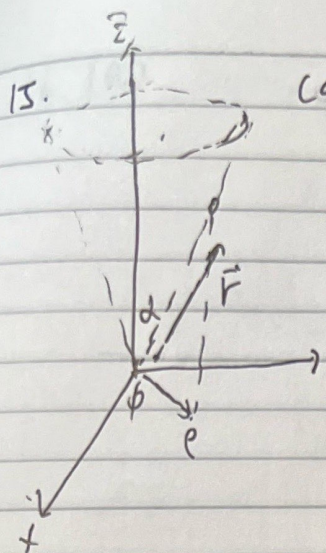
i.e. the tensions of ~~string~~ strings.

in this case, whereas the

Newtonian method takes the

forces of constraints into

account.



(a) In cylindrical coordinates:

P.E. $V = mgz$

K.E. $T = \frac{1}{2} m \underline{v} \cdot \underline{v}$

~~$\underline{v} = \dot{r} \hat{r} + r \dot{\phi} \hat{\phi} + \dot{z} \hat{z}$~~

$\underline{v} = \dot{r} \hat{r} + r \dot{\phi} \hat{\phi} + \dot{z} \hat{z}$

$\therefore T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2)$

Constraints: $\dot{\phi} = \omega = \text{const.}$ $\rho = r \sin \alpha$, $z = r \cos \alpha$

$\therefore L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2) - mgz$

$= \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2 \alpha \omega^2) - mgr \cos \alpha$

$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r} \Rightarrow$

~~$m \ddot{r} = m \omega^2 \sin^2 \alpha r - mg \cos \alpha$~~

$\ddot{r} = \omega^2 \sin^2 \alpha r - g \cos \alpha$

$\ddot{r} = -g$

When $\alpha \rightarrow 0$, ~~$\ddot{r} = -g$~~ , which corresponds to a bead on a vertical wire

When $\alpha \rightarrow \frac{\pi}{2}$, ~~$\ddot{r} = \omega^2 r$~~ , which corresponds to a bead on a horizontal wire

(b)

~~$\ddot{r}$~~ $\ddot{r} - \omega^2 \sin^2 \alpha r = -g \cos \alpha$

C.F. $r = A e^{\lambda t} + B e^{-\lambda t}$ where ~~$\lambda = \omega \sin \alpha$~~ $\lambda = \omega \sin \alpha$

P.I. Let $r = C$, $-\omega^2 \sin^2 \alpha (C) = -g \cos \alpha$

$\therefore C = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}$

$\therefore r(t) = A e^{+\omega \sin \alpha t} + B e^{-\omega \sin \alpha t} + \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}$

Initial Condition: $t=0$, $r=r_0$, $\dot{r}=0$

$$\therefore A+B + \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} = r_0$$

$$\omega \sin \alpha (A-B) + \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} = 0$$

$$\therefore A+B = r_0 - \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}$$

$$A-B = 0$$

$$\therefore A=B = \frac{r_0}{2} - \frac{g \cos \alpha}{2 \omega^2 \sin^2 \alpha}$$

$$\therefore r(t) = \left(\frac{r_0}{2} - \frac{g \cos \alpha}{2 \omega^2 \sin^2 \alpha} \right) \left[e^{\omega \sin \alpha t} + e^{-\omega \sin \alpha t} \right] + \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \quad (r > 0)$$

$$\text{For } r_0 = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}, \quad r(t) = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \text{ always.}$$

$\therefore$ The bead moves in circular ~~orbit~~ motion

when $r < r_0$, the bead accelerates towards the origin

when $r > r_0$, the bead accelerates outwards towards infinity.

(c) Angular momentum w.r.t origin I is not conserved due to the ~~torq~~ torque applied by the wire.

$$P_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r}$$

$$\text{Hamiltonian } H = P_r \dot{r} - L = \frac{1}{2} m (\dot{r}^2 - \omega^2 \sin^2 \alpha r^2) + m g r \cos \alpha$$

$$\text{Total Energy } T+V = \frac{1}{2} m \dot{r}^2 + \omega^2 \sin^2 \alpha r^2 + m g r \cos \alpha$$

$$\text{Clearly } H \neq T+V$$

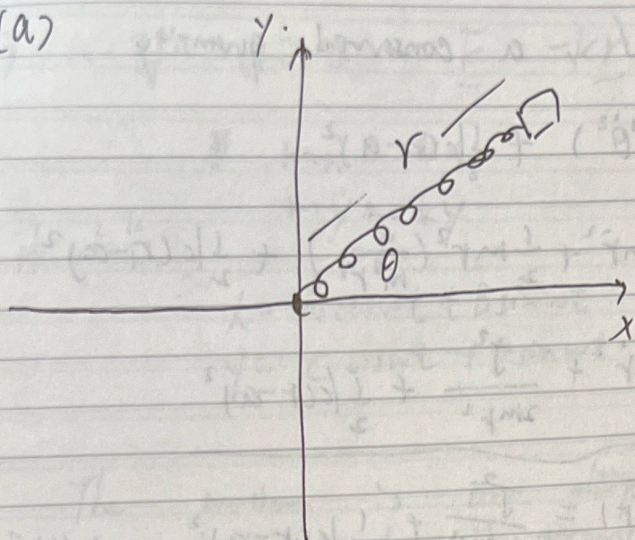
$$\therefore \frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0 \quad \therefore H \text{ is conserved}$$

$$T+V = H + m \omega^2 \sin^2 \alpha r^2$$

$$\cancel{T+V} = \cancel{H} \quad \therefore r = r(t) \text{ is a variable}$$

$$\therefore T+V \text{ is not conserved}$$

16. (a)



In polar coordinate :

$$\text{P.E. } V = \frac{1}{2} k(r-a)^2$$

$$\text{K.E. } T = \frac{1}{2} m \underline{v} \cdot \underline{v}$$

$$\underline{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}$$

$$\therefore T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$\therefore L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{1}{2} k(r-a)^2$$

L does not explicitly depend on θ
 $\therefore \theta$ is the cyclic coordinate

$$p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} = J \text{ is the conserved quantity}$$

$$\text{because } \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} = 0$$

This corresponds to the conservation of angular momentum

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r}$$

$$\therefore \boxed{m \ddot{r} = m \dot{\theta}^2 r - k(r-a)}$$

$$\therefore m r^2 \dot{\theta} = J \quad \therefore \dot{\theta} = \frac{J}{m r^2}$$

$$\dot{\theta}^2 = \frac{J^2}{m^2 r^4}$$

$$\therefore \boxed{m \ddot{r} = \frac{J^2}{m r^3} - k(r-a)}$$

(b) Total Energy $T+V$, a conserved quantity, is

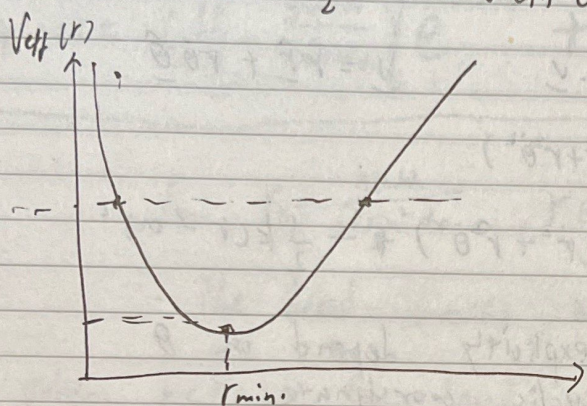
$$E = T+V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}k(r-a)^2$$

$$= \cancel{\frac{1}{2}m\dot{\theta}^2} \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\left(\frac{J^2}{m^2r^4}\right) + \frac{1}{2}k(r-a)^2$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + \frac{1}{2}k(r-a)^2$$

define $V_{\text{eff}}(r) = \frac{J^2}{2mr^2} + \frac{1}{2}k(r-a)^2$

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$$



This effective potential $V_{\text{eff}}(r)$ is a function of single minimum, and approaches $V \rightarrow \infty$ at both ends, (as $r \rightarrow 0$ and as $r \rightarrow \infty$)

$\therefore$ The system is always bounded.

~~Special Case~~ ~~When $J=0$~~ When $r=r_{\text{min}}$ the motion is circular

In this case $\frac{d}{dr} V_{\text{eff}}(r) = 0$

$$\therefore -\frac{J^2}{2mr^3} + k(r-a) = 0$$

$$\therefore \frac{J^2}{mr^3} = k(r-a) \quad \therefore \frac{m\dot{r}^2 r^2}{mr^3} = k(r-a)$$

$$\therefore \frac{m\dot{r}^2}{r} = k(r-a). \quad \text{This is consistent with}$$

Newton's laws⁹ because $k(r-a)$ is the spring force and $m\frac{\dot{r}^2}{r}$ is the centripetal force.

When they are balanced the mass ~~undergoes~~ undergoes a circular motion.
uniform

(c) If $a \approx 0$ $\underline{F} = -k\underline{r}$

$\therefore m\ddot{x} = -kx$

$m\ddot{y} = -ky$

if $\omega^2 = \frac{k}{m}$

$\therefore x = A \cos \omega t + B \sin \omega t$

$y = C \cos \omega t + D \sin \omega t$

The motion is an ellipse only when initial displacement is perpendicular to initial velocity. In this case:

$x = A \cos \omega t$

$y = D \sin \omega t$

initial

(we choose x to be the axis of displacement and y be the axis of velocity without the loss of generality).

$\frac{x^2}{A^2} + \frac{y^2}{D^2} = \cos^2 \omega t + \sin^2 \omega t = 1$

$\therefore$ This is an ellipse.

Energy equation becomes:

$E = \frac{1}{2} m \dot{r}^2 + \frac{J^2}{2mr^2} + \frac{1}{2} kr^2 \quad (a \approx 0)$

at major and minor axis $\dot{r} = 0$

$\therefore E = \frac{J^2}{2mr^2} + \frac{1}{2} kr^2$

$\therefore \frac{1}{2} kr^4 + \frac{J^2}{2m} - Er^2 = 0$

$\therefore r^2 = \frac{E \pm \sqrt{E^2 - \frac{kJ^2}{m}}}{k} = \frac{1 \pm \sqrt{1 - \frac{kJ^2}{mE^2}}}{\frac{k}{E}} = \frac{E}{k} (1 \pm \sqrt{1 - \frac{kJ^2}{mE^2}})$

let $\boxed{\epsilon = \sqrt{1 - \frac{kJ^2}{mE^2}}}$

$\epsilon^2 = 1 - \frac{kJ^2}{mE^2}$

$\therefore \frac{k}{E} = \frac{mE^2}{kJ^2} (1 - \epsilon^2)$

$$\therefore \frac{kJ^2}{mE^2} = 1 - e^2 \quad \therefore \sqrt{1 - e^2} = \frac{kJ}{mE}$$

$$\therefore \frac{mE^2}{kJ^2} = \frac{1}{1 - e^2} \quad \therefore \left(\frac{E}{k}\right) \left(\frac{mE}{J^2}\right) = \frac{1}{1 - e^2}$$

$$\therefore \frac{E}{k} = \frac{J^2}{mE} \left(\frac{1}{1 - e^2}\right)$$

here we know that

$$r = \frac{E}{k} (1 \pm e)$$

$$\therefore r^2 = \frac{J^2}{mE} \frac{1 \pm e}{1 - e^2}$$

$$\text{let } r_0^2 = \frac{J^2}{mE}$$

$$r^2 = r_0^2 \frac{1 \pm e}{(1 - e)(1 + e)} = \frac{r_0^2}{1 \mp e}$$

$\therefore$ major axis

$$a = \frac{r_0}{\sqrt{1 - e}}$$

minor axis

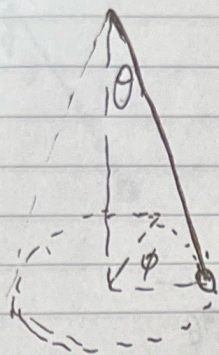
$$b = \frac{r_0}{\sqrt{1 + e}}$$

17. (a)

In spherical coordinates:

$$V = -mgd \cos \theta$$

z



$$T = \frac{1}{2} m \underline{v} \cdot \underline{v}$$

~~$$\underline{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + r \sin \theta \dot{\phi} \hat{\phi}$$~~

~~$$\underline{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + r \sin \theta \dot{\phi} \hat{\phi}$$~~

$$\underline{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + r \sin \theta \dot{\phi} \hat{\phi}$$

$$\therefore T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2)$$

constraints: $r = l$, $\dot{r} = 0$

~~$$\therefore T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2)$$~~

$$\therefore L = T - V = \boxed{\frac{1}{2} m (l^2 \dot{\theta}^2 + l^2 \sin^2 \theta \dot{\phi}^2) + mgd \cos \theta}$$

$$P_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

$$P_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = m l^2 \sin^2 \theta \dot{\phi}$$

$$H = P_{\theta} \dot{\theta} + P_{\phi} \dot{\phi} - L = \underbrace{\frac{1}{2} m (l^2 \dot{\theta}^2 + l^2 \sin^2 \theta \dot{\phi}^2)}_T - \underbrace{mgd \cos \theta}_V$$

$$= T + V \quad (\text{as required})$$

(b) L does not explicitly depend on ϕ

$$\frac{d}{dt} (P_{\phi}) = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} = 0$$

$\therefore P_{\phi} = m l^2 \sin^2 \theta \dot{\phi}$ is a conserved quantity.

This shows that ~~the~~ the angular momentum in the z -direction is conserved.

$$H = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m l^2 \sin^2 \theta \dot{\phi}^2 - mgl \cos \theta$$

$$\therefore \frac{d}{dt} (m l^2 \sin^2 \theta \dot{\phi}) = 0$$

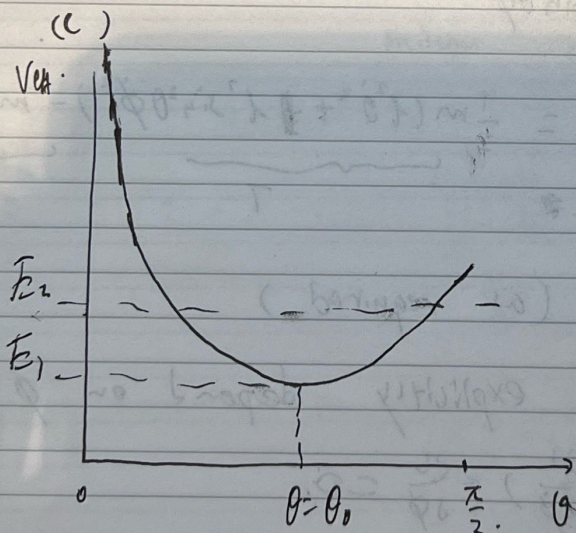
$$\therefore m l^2 \sin^2 \theta \dot{\phi} = J_z$$

$$\therefore \dot{\phi} = \frac{J_z}{m l^2 \sin^2 \theta} \quad \therefore \dot{\phi}^2 = \frac{J_z^2}{m^2 l^4 \sin^4 \theta}$$

$$\therefore H = \frac{1}{2} m l^2 \dot{\theta}^2 + \underbrace{\frac{J_z^2}{2 m l^2 \sin^2 \theta}}_{\text{call this } V_{\text{eff}}(\theta)} - mgl \cos \theta$$

$$\therefore H = \frac{1}{2} m l^2 \dot{\theta}^2 + V_{\text{eff}}(\theta)$$

$$V_{\text{eff}}(\theta) = \frac{J_z^2}{2 m l^2 \sin^2 \theta} - mgl \cos \theta \quad J_z \text{ is constant.}$$



$$\therefore L = \frac{1}{2} m l^2 \dot{\theta}^2 + l^2 \sin^2 \theta \dot{\phi}^2 + mgl \cos \theta$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta}$$

$$\therefore m l^2 \ddot{\theta} = m l^2 \dot{\phi}^2 \sin \theta \cos \theta - mgl \sin \theta \quad (1)$$

The shape of $V_{\text{eff}}(\theta)$ shows that if $\theta = \theta_0$, the system is bound to $\theta = \theta_0$. i.e. θ cannot change because its total energy is not high enough.

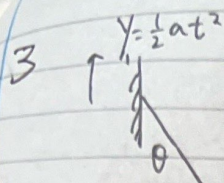
(E_1 corresponds to $\theta = \theta_0$, the circular motion case, and E_2 corresponds to the up and down case).

$\therefore$ The pendulum can move with ^{uniform} circular motion (θ does not change).

In this case $\theta = \theta_0$, $\dot{\theta} = 0$. ~~Subst~~ Substitute this into (1) and set $\dot{\phi} = \Omega$ we get

$$0 = m l^2 \Omega^2 \sin \theta \cos \theta - mgl \sin \theta$$

$$\therefore \boxed{\Omega = \frac{g}{l \cos \theta_0}}$$



$$x = l \sin \theta \quad y = \frac{1}{2} at^2 - l \cos \theta$$

$$\dot{x} = \cancel{t \sin \theta} l \cos \theta \dot{\theta} \quad \dot{y} = at + l \sin \theta \dot{\theta}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (l^2 \cos^2 \theta \dot{\theta}^2 + a^2 t^2 + 2at l \sin \theta \dot{\theta} + l^2 \sin^2 \theta \dot{\theta}^2)$$

$$\therefore T = \frac{1}{2} m (l^2 \dot{\theta}^2 + a^2 t^2 + 2at l \sin \theta \dot{\theta})$$

$$V = mgy = \frac{1}{2} m g a t^2 - m g l \cos \theta$$

$$\therefore L = T - V = \frac{1}{2} m (l^2 \dot{\theta}^2 + a^2 t^2 + 2at l \sin \theta \dot{\theta}) - \frac{1}{2} m g a t^2 + m g l \cos \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} + m a t l \sin \theta \quad \frac{\partial L}{\partial \theta} = m a t l \cos \theta - m g l \sin \theta$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} \Rightarrow m l^2 \ddot{\theta} + m a l (\sin \theta + t \cos \theta \dot{\theta}) = m a t l \cos \theta - m g l \sin \theta$$

$$\therefore m l^2 \ddot{\theta} + m a l \sin \theta + m g l \sin \theta = 0$$

$$\therefore \ddot{\theta} + \frac{g+a}{l} \sin \theta = 0$$

For small angles $\theta \approx \sin \theta$

$$\therefore \ddot{\theta} + \frac{g+a}{l} \theta = 0 \quad \therefore \omega = \sqrt{\frac{g+a}{l}}$$

$\therefore$ period

$$P = 2\pi \sqrt{\frac{l}{g+a}}$$