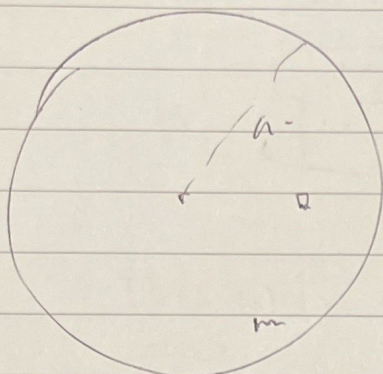


Mechanics 5

Ziyan Li



$$M = \int dm. \quad \text{Surface density } \sigma = \frac{m}{\pi a^2}$$

$$dm = \sigma dA$$

$$\therefore I = \int r^2 dm = \int r^2 dA$$

$$= \int_0^{2\pi} \int_0^a \sigma r^3 dr d\theta = 2\pi \left(\frac{a^4}{4}\right) \sigma$$

$$= 2\pi \left(\frac{a^4}{4} \times \frac{m}{\pi a^2}\right) = \boxed{\frac{1}{2} m a^2}$$

$$\underline{J} = I\omega = \boxed{\frac{1}{2} m a^2 \omega}$$

$$T = \frac{1}{2} I \omega^2 = \boxed{\frac{1}{4} m a^2 \omega^2}$$

Moment of Inertia I' wrt of combined mass is $I' = \frac{1}{2} (2m) a^2 = m a^2$.

$$\Sigma J = J_A + J_B$$

$$J_A = \frac{1}{2} m a^2 \omega$$

$$J_B = 0$$

$$\Sigma J = J_A + J_B = \frac{1}{2} m a^2 \omega$$

This is ~~conserved~~ conserved.

$$\therefore \frac{1}{2} m a^2 \omega = J' = (m a^2) \omega'$$

$$\therefore \boxed{\omega' = \frac{1}{2} \omega}$$

$$\boxed{J' = \frac{1}{2} m a^2 \omega}$$

$$\cancel{T' = \frac{1}{2} (2m) (\omega')^2 = \frac{1}{2} (2m) \left(\frac{1}{2} \omega\right)^2 =}$$

$$T' = \frac{1}{2} (m a^2) \left(\frac{1}{4} \omega^2\right) = \boxed{\frac{1}{8} m a^2 \omega^2}$$

(2) (i) $I = \frac{1}{2}Mr^2$ $\frac{\Delta J}{\Delta t} = N$
 $Nt = I\omega_0 - I\omega_1 = \frac{1}{2}Mr^2(\omega_0^2 - \omega_1^2)$

$\therefore N = \frac{Mr^2(\omega_0^2 - \omega_1^2)}{2t}$

(ii) $N = \frac{dJ}{dt}$ let $N = -k\omega$

$\therefore \frac{dJ}{dt} = -k\omega$

$\therefore \frac{1}{2}Mr^2 \frac{d\omega}{dt} = -k\omega$

$\therefore \frac{d\omega}{dt} = -\frac{2k}{Mr^2} \omega$

$\therefore \int_{\omega_0}^{\omega_1} \frac{d\omega}{\omega} = -\int_0^t \frac{2k}{Mr^2} dt \quad \therefore \ln \frac{\omega_1}{\omega_0} = -\frac{2k}{Mr^2} t$

$\therefore \frac{2k}{Mr^2} t = \ln \frac{\omega_0}{\omega_1}$

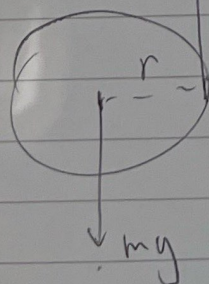
$\therefore k = \frac{Mr^2}{2t} \ln \frac{\omega_0}{\omega_1}$

$|N| = k\omega = \frac{Mr^2 \omega}{2t} \ln \frac{\omega_0}{\omega_1}$

Initially, let $a = \ddot{x}$ $\alpha = \ddot{\theta} = \frac{\ddot{x}}{r} = \frac{a}{r}$

(3)

Initially,



$I = \frac{1}{2}mr^2$

$\Sigma F = mg - T = ma$

$\Sigma \tau = Tr = I\alpha = \frac{Ia}{r} = \frac{1}{2}mr^2 \frac{a}{r} = \frac{1}{2}mra$

$\therefore T = \frac{1}{2}ma$

$\therefore mg - \frac{1}{2}ma = ma \quad \therefore \frac{3}{2}a = g$

$\therefore a = \frac{2}{3}g$

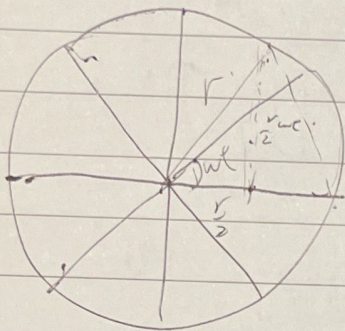
$T = \frac{1}{2}ma = \frac{1}{2}(m)(\frac{2}{3}g) = \frac{1}{3}mg$

(4) (a) Centrifugal force = Mg $I = Mr^2$

$\therefore \omega^2 r m = mg \therefore \omega^2 = \frac{g}{r} \therefore \omega = \sqrt{\frac{g}{r}}$

$W = \Delta T = \frac{1}{2} I \omega^2 = \frac{1}{2} (Mr^2) \left(\frac{g}{r}\right) = \boxed{\frac{1}{2} Mgr}$

(b)



$I_1 \omega_1 = I_2 \omega_2$ $I_1 = (M+m)r^2$

$I_2 = Mr^2$

Conservation of Angular momentum

$(M+m)r^2 \omega_1 = Mr^2 \omega_2$

$\therefore \frac{\omega_2}{\omega_1} = \frac{M+m}{M}$

fractional change in apparent gravity is

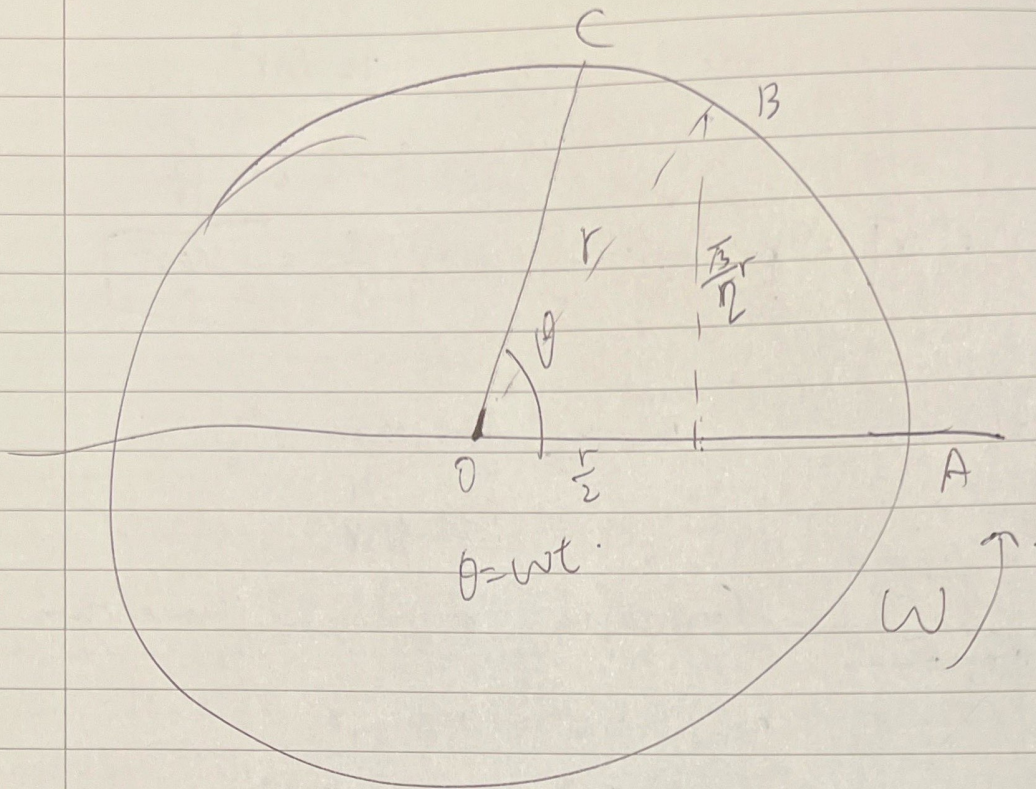
$\frac{\Delta g}{g} = \frac{\omega_2^2 r - \omega_1^2 r}{\omega_1^2 r} = 1 - 1 + \frac{\omega_2^2}{\omega_1^2} = \frac{M+m}{M}$

(c) When he is half-way, the angular velocity is ω_3

$\frac{1}{2} (M+m)r^2 \omega_1^2 = (Mr^2 + m(\frac{r}{2})^2) \omega_3^2 = (M + \frac{m}{4}) \omega_3^2 r^2$

$\therefore \frac{\omega_3}{\omega_1} = \frac{M + \frac{m}{4}}{M+m}$

Velocity at release $v_3 = \omega_3 \frac{r}{2} = \frac{M + \frac{m}{4}}{M+m} r \omega_1$



The angular momentum of neither the astronaut nor the cylinder changes after release.

$$\frac{\sqrt{3}}{2}r = \omega \left(\frac{r}{2}\right)t \quad \therefore \omega t = \sqrt{3}r$$

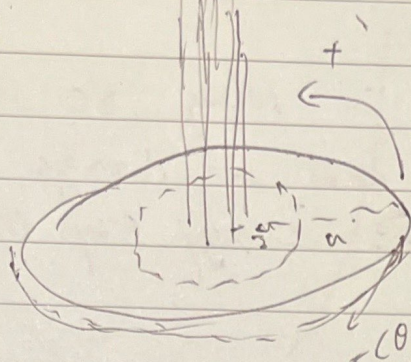
$$\therefore |\widehat{AC}| = \omega t r = \sqrt{3}r$$

$$\because \angle BOA = 60^\circ = \frac{\pi}{3} \quad \therefore |\widehat{AB}| = \frac{\pi}{3}r$$

$$\therefore \widehat{BC} = \widehat{AC} - \widehat{AB} = \sqrt{3}r - \frac{\pi}{3}r = \left(\sqrt{3} - \frac{\pi}{3}\right)r$$

$\text{mvr} = \frac{1}{2} m v^2$
 $\text{I} \dot{\theta} = \frac{1}{2} m v^2$

(5)



$$I = \frac{1}{2} m a^2$$

$$\Sigma \tau = I \ddot{\theta}$$

$$I \ddot{\theta} = -C \theta$$

$$\ddot{\theta} = -\frac{C}{I} \theta$$

$$\text{let } \omega^2 = \frac{C}{I} = \frac{2C}{m a^2}$$

$$\therefore \theta = A \cos(\omega t + \phi)$$

$\therefore$ The oscillation is simple harmonic and

$$T = 2\pi \sqrt{\frac{m a^2}{2C}}$$

(a)

$$I' = \frac{1}{2} m a^2 + m \left(\frac{a}{2}\right)^2 = \frac{1}{2} m a^2 + \frac{1}{4} m a^2 = \frac{3}{4} m a^2$$

$$\therefore T' = 2\pi \sqrt{\frac{I'}{C}} = 2\pi \sqrt{\frac{\frac{3}{4} m a^2}{C}} = 2\pi \sqrt{\frac{3}{4} \frac{m a^2}{C}} \text{ for both (i) and (ii)}$$

(b) (i)

If it is dropped at the end of the string, angular momentum before and after are both zero. The combined mass starts moving at the original maximum height of the original mass with zero initial velocity.

$\therefore$ Amplitude does not change.

(ii) ~~Energy~~ Potential Energy stored in the torsional wire is $V = -\int_0^\theta -C \theta' d\theta' = \frac{1}{2} C \theta^2$.

At the midpoint of swing:

Conservation of Angular momentum $\Delta J = 0 \Rightarrow W_{\max} I = W'_{\max} I' \therefore \frac{W_{\max}}{W'_{\max}} = \frac{I}{I'} = \frac{1/2}{3/4} = \frac{2}{3}$

Conservation of energy $\Delta E = 0 \Rightarrow \frac{1}{2} C \theta_{\max}^2 = \frac{1}{2} C \theta'^2 \therefore \frac{1}{2} C \theta_{\max}^2 = \frac{1}{2} I' W_{\max}'^2$

$\therefore$ Ratio of Amplitudes $\frac{\theta'_{\max}}{\theta_{\max}} = \left(\frac{I W_{\max}}{I' W'_{\max}} \right)^{1/2} = \left(\frac{2}{3} \right)^{1/2} = \sqrt{\frac{2}{3}}$

(c) (i) total energy $E = \frac{1}{2} C \theta_{\max}^2$ $\therefore$ Adding the ring at the end of string does not change the value of $\theta_{\max}$ $\therefore$ ~~Energy~~ Energy stays the same.

(ii)

$$E = \frac{1}{2} I \omega_{\max}^2 = \frac{1}{2} J \omega_{\max}^2$$

$$E' = \frac{1}{2} I' \omega_{\max}'^2 = \frac{1}{2} J \omega_{\max}^2$$

$$\therefore \frac{\omega_{\max}'}{\omega_{\max}} = \frac{2}{3}$$

$$\boxed{\frac{E'}{E} = \frac{2}{3}}$$

(6) ~~I_0~~ $I_0 = \frac{1}{2} m a^2$

$$\therefore I_{O'} = \frac{1}{2} m a^2 + m x^2$$

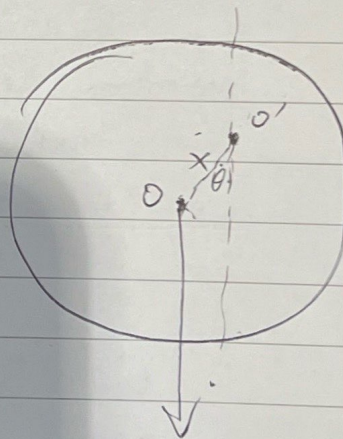
$$\Sigma \tau = -m g x \sin \theta = I_{O'} \ddot{\theta}$$

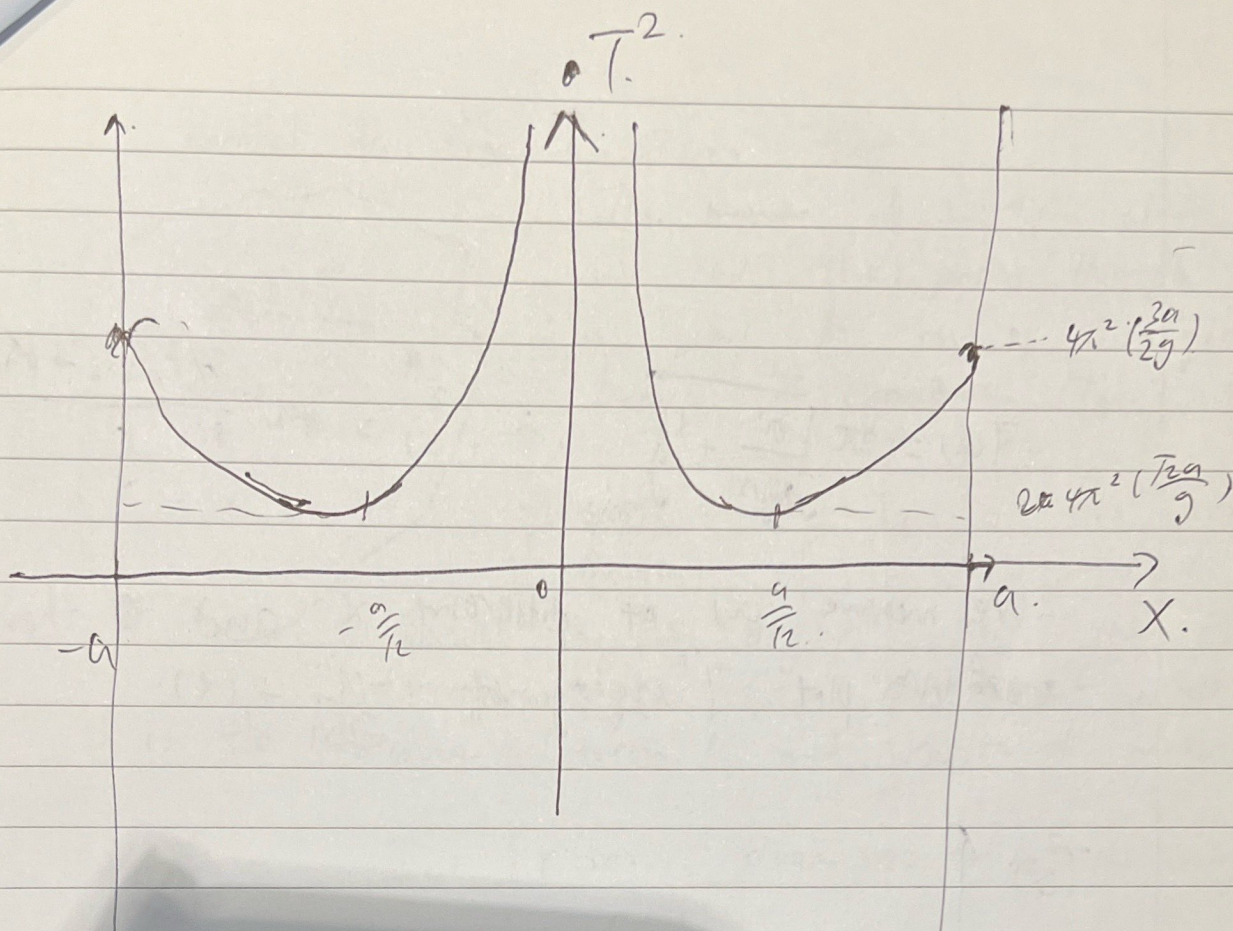
for small $\sin \theta \approx \theta$

$$\therefore \ddot{\theta} = \frac{-m g x}{I_{O'}} \theta$$

$$\therefore \theta = A \cos(\omega t + \phi) \quad \left[\omega^2 = \frac{m g |x|}{I_{O'}} \right]$$

$$\therefore T = 2\pi \sqrt{\frac{I}{m g |x|}} = 2\pi \sqrt{\frac{\frac{1}{2} m a^2 + m x^2}{m g |x|}} = \boxed{2\pi \sqrt{\frac{a^2}{2 g |x|} + \frac{|x|}{g}}}$$





$$T^2 = 254\pi^2 \left(\frac{a^2}{2gx} + \frac{|x|}{g} \right)$$

$$\frac{a^2}{2gx} + \frac{|x|}{g} = \frac{a^2}{2gx} - \frac{a^2}{2gx} + \frac{|x|}{g} + \frac{2a^2}{g^2} = \left(\frac{a}{\sqrt{2gx}} - \sqrt{\frac{|x|}{g}} \right)^2 + \frac{2a^2}{g^2}$$

$\therefore$ When $\frac{a}{\sqrt{2gx}} - \sqrt{\frac{|x|}{g}} = 0$, T^2 has ~~max~~ minimum value.

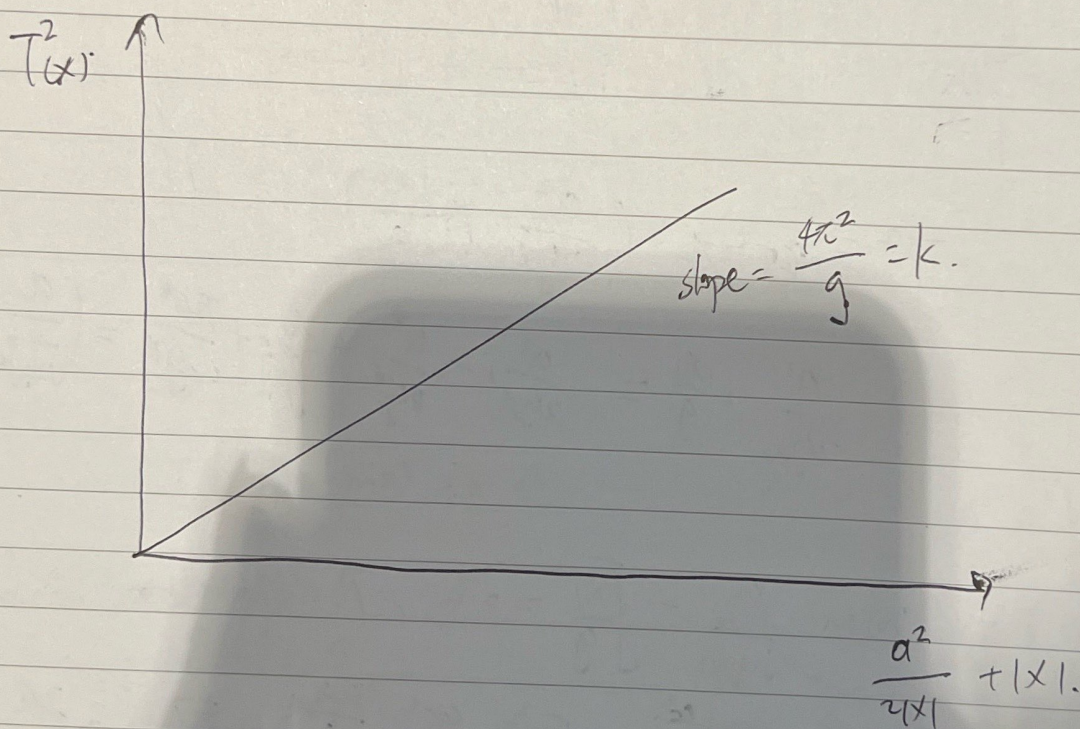
$$\cancel{T^2 = 4\pi^2 \frac{a^2}{g}} \quad T_{\min}^2 = 4\pi^2 \left(\frac{2a}{g} \right) \Rightarrow T_{\min} = 2\pi \sqrt{\frac{2a}{g}}$$

This happens when

$$\frac{a}{\sqrt{2gx}} = \frac{\sqrt{|x|}}{\sqrt{g}} \Rightarrow |x| = \frac{a^2}{2g} \Rightarrow \boxed{x = \pm \frac{a}{\sqrt{2g}}}$$

$$T(x) = 2\pi \sqrt{\frac{a^2}{2g|x|} + \frac{|x|}{g}} \quad \therefore T^2(x) = 4\pi^2 \frac{a^2/2|x| + |x|}{g}$$

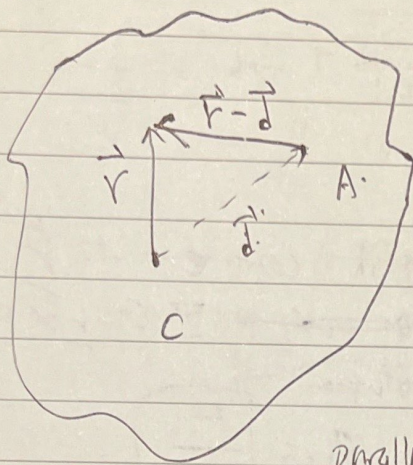
$\therefore$ we measure $T(x)$ at different x and then.
~~and~~ we plot $T^2(x)$ against $\frac{a^2}{2|x|} + |x|$.



The slope is $k = \frac{4\pi^2}{g}$

$$\therefore \boxed{g = \frac{k}{4\pi^2}}$$

7. Parallel Axis Theorem states that



The moment of inertia about an axis that passes through a point A on of an rigid body is ~~equal to~~ $I_A = I_C + md^2$, given by

where I_C is the moment of inertia of an axis that is parallel to the original axis that passes through the centre of mass. And d is the distance between A and centre of mass C.

Proof: $I_C = \int r^2 dm$ choose C to be the origin.

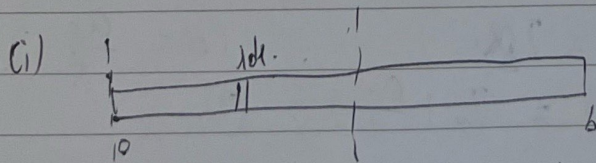
$$\text{At } \vec{r}_C = 0 = \frac{\int \vec{r} dm}{\int dm} \Rightarrow \int \vec{r} dm = 0.$$

$\therefore \vec{d}$ is a constant vector

$$\therefore I_A = \int (\vec{r} - \vec{d})^2 dm = \int r^2 dm - 2 \int \vec{r} \cdot \vec{d} dm + \int d^2 dm$$

$$= \int r^2 dm + d^2 \int dm - 2 \vec{d} \cdot \int \vec{r} dm$$

$$= I_C + md^2 + 0 = \underline{I_C + md^2}$$



$$\lambda = \frac{m}{b}$$

$$dm = \lambda dl$$

$$I_A = \int l^2 dm = \int_0^b \lambda l^2 dl = \lambda \frac{l^3}{3} \Big|_0^b = \frac{b^3 \lambda}{3} = \frac{b^3}{3} \left(\frac{m}{b} \right)$$

$$= \frac{1}{3} mb^2$$

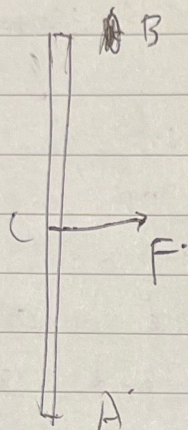
$$I_C = \int_{-b/2}^{b/2} \lambda l^2 dl = \frac{\lambda l^3}{3} \Big|_{-b/2}^{b/2} = \frac{\lambda b^3}{12} = \frac{b^3}{12} \left(\frac{m}{b} \right) = \frac{1}{12} mb^2$$

$$d = m\left(\frac{b}{2}\right)^2 = \frac{1}{4}mb^2$$

$$I_A = \frac{1}{3}mb^2 \quad I_C + md^2 = \frac{1}{12}mb^2 + \frac{1}{4}mb^2 = \frac{1}{3}mb^2$$

$$\therefore I_A = I_C + md^2$$

(a)

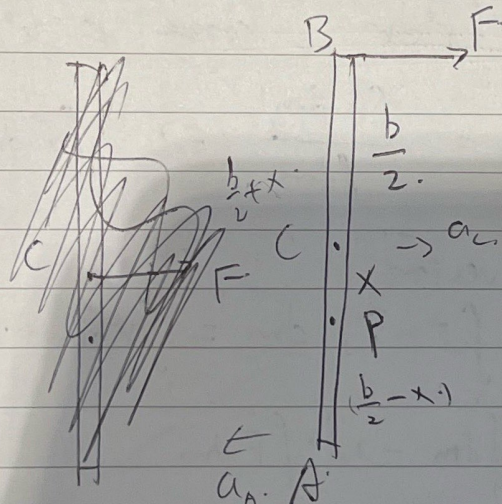


$\therefore F$ through centre $\therefore \vec{r} \Rightarrow$
 $\therefore \cancel{\sum \tau = 0} \quad \sum \tau = \vec{r} \times \vec{F} = 0$

$\therefore$ No rotation

$$\therefore a_A = a_C = \boxed{\frac{F}{m}}$$

(b)



(instantaneously stationary)

let axis of rotation be P and let $PC = x$.

$$I_P = \frac{1}{12}mb^2 + mx^2$$

Newton's second law for translation:

$$\sum F = ma$$

$$\therefore F = ma_c \quad (1)$$

for rotation:

$$\sum \tau = I\alpha$$

$$(\alpha = \frac{a_c}{x})$$

$$\therefore F\left(\frac{b}{2} + x\right) = I_P \left(\frac{a_c}{x}\right) \quad (2)$$

$$(1), (2) \Rightarrow F\left(\frac{b}{2} + x\right) = \left(\frac{1}{12}mb^2 + mx^2\right) \left(\frac{F}{mx}\right)$$

$$\therefore \frac{1}{2}Fb + Fx = \frac{Fb^2}{12x} + Fx$$

$$\therefore \frac{1}{2} = \frac{b}{12x}$$

$$x = \frac{b}{6}$$

$$x = \frac{b}{6}$$

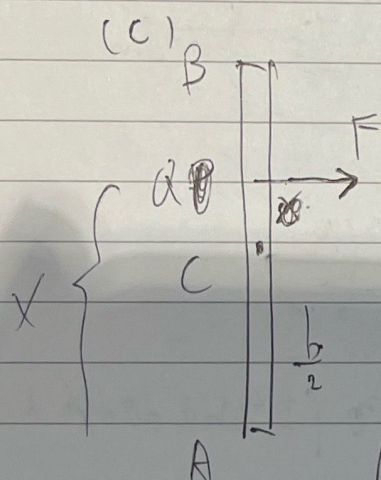
$$\therefore AP = \frac{b}{2} - \frac{b}{6} = \frac{b}{3}$$

$$\alpha = \frac{a_A}{AP} = \frac{a_c}{CP} \quad \therefore a_A = \frac{AP}{CP} a_c = \frac{b/3}{b/6} \left(\frac{F}{m} \right) = \frac{2F}{m}$$

$\therefore$ P temporarily ~~stop~~ stationary

$\therefore$ ~~A starts~~ A starts to move in the opposite direction of B and centre of mass

$$\therefore \boxed{a_A = -\frac{2F}{m}}$$



let the point where force is applied be D. and let $CD = x$.

A is the instantaneously stationary axis of rotation.

$$I_A = \frac{1}{12}mb^2 + m\left(\frac{b}{2}\right)^2 \quad I_A = \frac{1}{3}mb^2$$

to Newton's Second Law for translation :

$$\Sigma F = ma_c \Rightarrow F = ma_c \Rightarrow a_c = \frac{F}{m}$$

rotation:

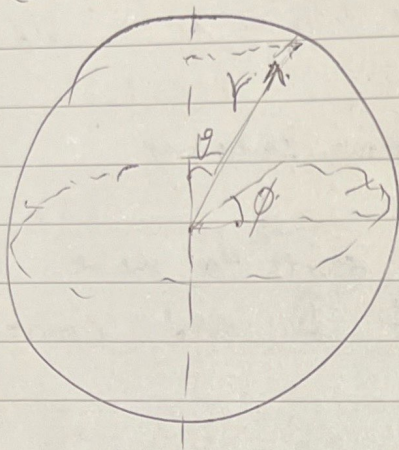
$$\Sigma \tau = I\alpha \Rightarrow F\left(x + \frac{b}{2}\right) = I_A \alpha$$

$$Fx = I_A \frac{a_c}{x} \quad \therefore Fx = \frac{1}{3}mb^2 \alpha$$

$$\alpha = \frac{a_c}{\frac{b}{2}} = \frac{2a_c}{b} = \frac{2F}{mb} \quad \therefore Fx = \frac{1}{3}mb^2 \left(\frac{2F}{mb} \right) = \frac{2}{3}Fb$$

$$\therefore \boxed{x = \frac{2}{3}b}$$

(8) (a) $\vec{r}$ hollow



~~$dS = R^2 \sin \theta d\theta d\phi$~~

$$dS = R^2 \sin \theta d\theta d\phi = R^2 \sin \theta d\theta d\phi$$

surface density $\sigma = M / (4\pi R^2)$

$$dm = \sigma dS \quad l = R \sin \theta$$

$$\therefore I = \int \int_{\text{shell}} R^2 \sigma dS$$

$$= \int_0^{2\pi} \int_0^\pi \sigma R^4 \sin \theta d\theta d\phi$$

$$= \int_0^{2\pi} \sigma R^4 \cos \theta \Big|_0^\pi d\phi$$

$$I = \int l^2 dm$$

$$\therefore I = \int \int_{\text{shell}} R^2 \sin^2 \theta \cdot \sigma dS = \int_0^{2\pi} \int_0^\pi$$

$$I = \int \int_{\text{shell}} l^2 dm = \int \int_{\text{shell}} \sigma R^2 \sin^2 \theta (R^2 \sin \theta d\theta d\phi)$$

$$= \int_0^{2\pi} \int_0^\pi \sigma R^4 \sin^3 \theta d\theta d\phi = \sigma R^4 \int_0^{2\pi} \int_0^\pi \sin \theta (1 - \cos^2 \theta) d\theta d\phi$$

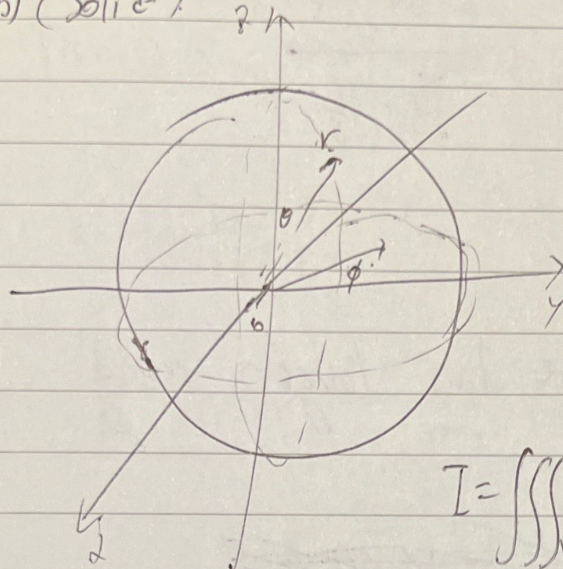
$$= \sigma R^4 \int_0^{2\pi} \left[\int_0^\pi \sin \theta d\theta + \int_0^\pi \sin \theta \cos^2 \theta d\theta \right] d\phi$$

$$= \sigma R^4 \int_0^{2\pi} \left[-\cos \theta \Big|_0^\pi + \left[\frac{\cos^3 \theta}{3} \right]_0^\pi \right] d\phi$$

$$= \sigma R^4 \int_0^{2\pi} 2\pi R^4 \left(2 - \frac{2}{3} \right) d\phi$$

$$= 2\pi \left(\frac{M}{4\pi R^2} \right) R^4 \left(\frac{4}{3} \right) = \boxed{\frac{2}{3} M R^2}$$

(b) (Solid)



$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

$$dm = \rho dV$$

$$l = r \sin \theta$$

$$dV = r^2 \sin \theta dr d\theta d\phi$$

$$I = \iiint_V l^2 dm = \int_0^{2\pi} \int_0^\pi \int_0^R \rho r^4 \sin^3 \theta dr d\theta d\phi$$

$$= \rho \int_0^{2\pi} \int_0^\pi \left[\frac{r^5}{5} \right]_0^R \sin^3 \theta d\theta d\phi$$

$$\left(\int_0^\pi \sin^3 \theta d\theta = \frac{4}{3} \text{ from (a)} \right)$$

$$= \frac{\rho R^5}{5} \int_0^{2\pi} \frac{4}{3} d\phi = \frac{2\pi}{5} \rho R^5 \left(\frac{4}{3} \right)$$

$$= \frac{2\pi}{5} \left(\frac{4}{3} \right) \left(\frac{3M}{4\pi R^3} \right) R^5 = \boxed{\frac{2}{5} MR^2}$$

(c)

Rolling without slipping $\Rightarrow V = \omega R$

Conservation of Energy

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

$$\therefore \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2 = mgL \sin \theta$$

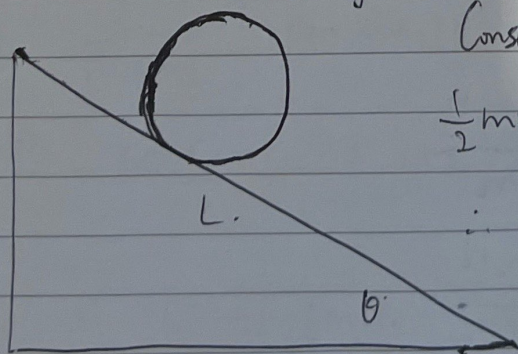
$$\therefore \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} = 2mgL \sin \theta$$

$$\text{let } \sqrt{gL \sin \theta} \text{ be } C$$

$$\therefore V = \sqrt{\frac{2mgL \sin \theta}{m + \frac{I}{R^2}}}$$

$$V_1 = V(I = \frac{2}{3}MR^2) = \sqrt{\frac{2mgL \sin \theta}{\frac{5}{3}M}} = \sqrt{\frac{6}{5}} C$$

$$V_2 = V(I = \frac{3}{5}MR^2) = \sqrt{\frac{2mgL \sin \theta}{\frac{8}{5}M}} = \sqrt{\frac{10}{8}} C$$



$$L \sin \theta$$

$$L$$

$$\theta$$

$$\frac{2}{3} + 1$$

$$\frac{2}{3}$$

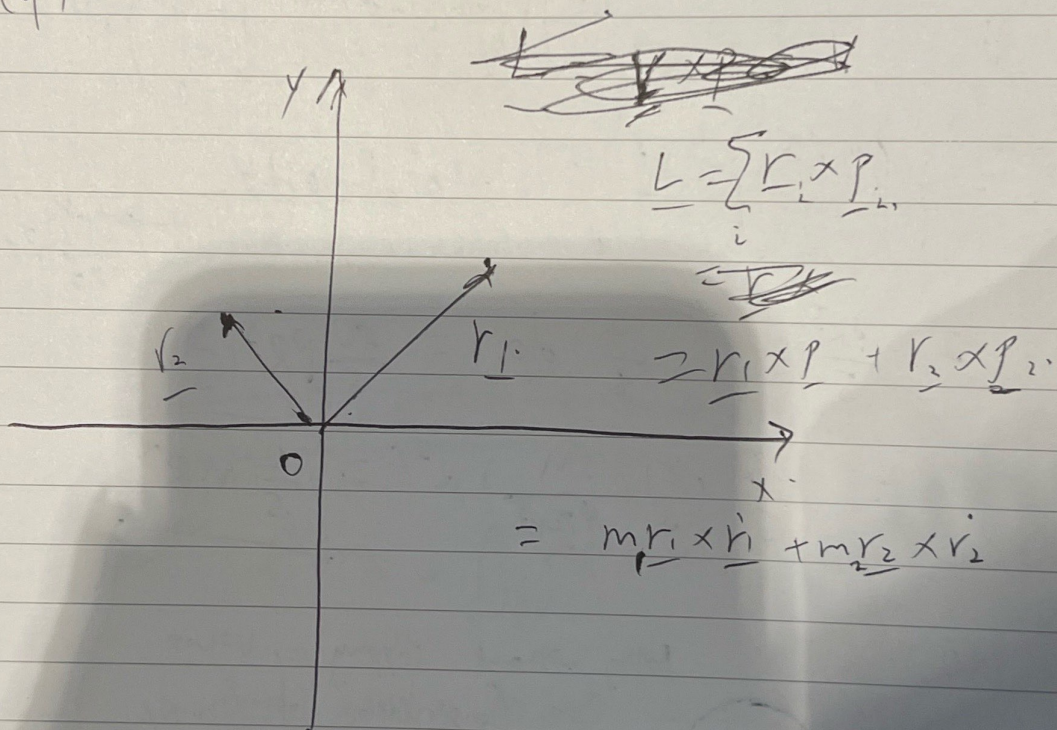
$$\frac{2}{3} + 1$$

$$\therefore \frac{\Delta v}{\langle v \rangle} = \frac{2(W_2 - v_1)}{v_2 + v_1} = \frac{2(\sqrt{\frac{10}{9}} - \sqrt{\frac{6}{5}}) \text{ K}}{\sqrt{\frac{10}{9}} + \sqrt{\frac{6}{5}} \text{ K}} = 0.08712$$

$$\approx \boxed{8.7\%}$$

The solid sphere has larger speed

(9).



$$\underline{r}_{cm} = \frac{m_1 \underline{r}_1 + m_2 \underline{r}_2}{m_1 + m_2}$$

$$\underline{\dot{r}}_{cm} = \frac{m_1 \underline{\dot{r}}_1 + m_2 \underline{\dot{r}}_2}{m_1 + m_2}$$

$$\underline{r} = \underline{r}_1 - \underline{r}_2$$

$$\underline{\dot{r}} = \underline{\dot{r}}_1 - \underline{\dot{r}}_2$$

$$N = \frac{m_1 m_2}{m_1 + m_2}$$

$$(A+B) \times (C+D) \\ = A \times C + A \times D + B \times C + B \times D$$

$$(m_1 + m_2) \underline{r_{cm}} \times \underline{\dot{r}_{cm}} \\ = (m_1 + m_2) \left(\frac{m_1 \underline{r_1} + m_2 \underline{r_2}}{m_1 + m_2} \right) \times \left(\frac{m_1 \underline{\dot{r}_1} + m_2 \underline{\dot{r}_2}}{m_1 + m_2} \right) \\ = \frac{1}{(m_1 + m_2)} (m_1^2 \underline{r_1} \times \underline{\dot{r}_1} + m_1 m_2 \underline{r_2} \times \underline{\dot{r}_1} + m_1 m_2 \underline{r_1} \times \underline{\dot{r}_2} + m_2^2 \underline{r_2} \times \underline{\dot{r}_2})$$

$$\underline{L} = \frac{m_1 m_2}{m_1 + m_2} (\underline{r_1} - \underline{r_2}) \times (\underline{\dot{r}_1} - \underline{\dot{r}_2}) \\ = \frac{m_1 m_2}{m_1 + m_2} (\underline{r_1} \times \underline{\dot{r}_1} - \underline{r_2} \times \underline{\dot{r}_1} - \underline{r_1} \times \underline{\dot{r}_2} + \underline{r_2} \times \underline{\dot{r}_2})$$

$$\therefore (m_1 + m_2) \underline{r_{cm}} \times \underline{\dot{r}_{cm}} + \underline{L} \\ = \frac{1}{m_1 + m_2} [m_1^2 \underline{r_1} \times \underline{\dot{r}_1} + m_1 m_2 \underline{r_2} \times \underline{\dot{r}_1} + m_1 m_2 \underline{r_1} \times \underline{\dot{r}_2} \\ + m_2^2 \underline{r_2} \times \underline{\dot{r}_2} + m_1 m_2 \underline{r_1} \times \underline{\dot{r}_1} - m_1 m_2 \underline{r_2} \times \underline{\dot{r}_1} - m_1 m_2 \underline{r_1} \times \underline{\dot{r}_2} \\ + m_1 m_2 \underline{r_2} \times \underline{\dot{r}_2}] \\ = \frac{1}{m_1 + m_2} [(m_1 + m_2) m_1 \underline{r_1} \times \underline{\dot{r}_1} + (m_1 + m_2) m_2 \underline{r_2} \times \underline{\dot{r}_2}] \\ = m_1 \underline{r_1} \times \underline{\dot{r}_1} + m_2 \underline{r_2} \times \underline{\dot{r}_2} = \underline{L}$$

$$\therefore \underline{L} = (m_1 + m_2) \underline{r_{cm}} \times \underline{\dot{r}_{cm}} + \underline{L}$$

10 (a) ~~$I = \frac{1}{12} ml^2$~~

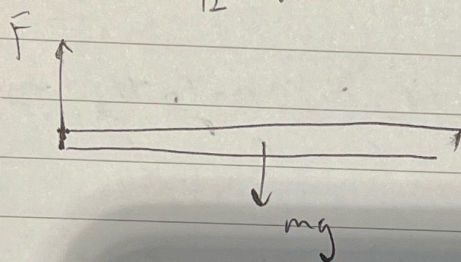
The motion of the bar is a combination of the free fall motion of the centre of mass and its rotation with constant angular velocity relative to the centre of mass.

(b) ~~$I = \frac{1}{12} ml^2$~~ ~~$I = \frac{1}{12} ml^2$~~

~~No net force~~ No net force on the bar during motion at

$\therefore$ CM is the centre of rotation.

$$\therefore I = \frac{1}{12} ml^2$$



At the instant of blow:

Newton's laws:

Translation: $(F - mg) \Delta t = m \Delta v$

$\therefore F \rightarrow \infty, \Delta t \rightarrow 0 \quad \therefore mg \Delta t \rightarrow 0$

$\therefore \Delta v = \frac{F \Delta t}{m} \Rightarrow F \Delta t = m \Delta v$

Rotation: $I \omega_0 = F \left(\frac{l}{2} \right) \Delta t \Rightarrow F \Delta t = \frac{2 I \omega_0}{l} = \frac{m l \omega_0}{6}$

$\therefore \Delta v = \frac{\omega_0 l}{6}$

$\therefore H_{\max} = \frac{\Delta v^2}{2g} =$

~~$m^2 \omega_0^2$~~ $\frac{\omega_0^2 l^2}{72g}$

time to reach maximum height :

$$t' = \frac{\Delta v}{g} = \boxed{\frac{u \sin \theta}{g}}$$

(c) $\therefore$ time to come back to original position. for cm

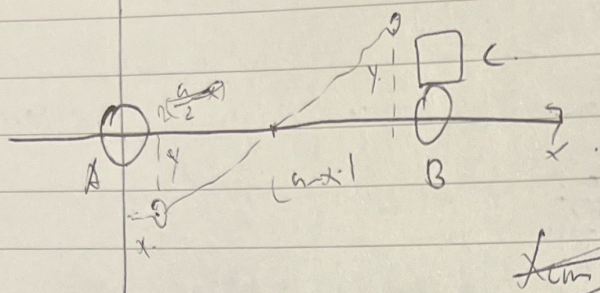
$$t = 2t' = \frac{u \sin \theta}{g}$$

If the bar is in its original position when cm comes back to original height, then,

$$u \sin \theta = 2n\pi \quad \therefore t = \frac{2n\pi}{u \sin \theta}$$

$$\therefore t^2 = t \times t = \left(\frac{2n\pi}{u \sin \theta} \right) \left(\frac{u \sin \theta}{g} \right) = \boxed{\frac{2n\pi^2}{g}}$$

(11) (a)



$$\underline{r}_{cm} = \frac{m_A \underline{r}_A + m_B \underline{r}_B + m_C \underline{r}_C}{m_A + m_B + m_C}$$

$$= \frac{\underline{r}_A + \underline{r}_B + \underline{r}_C}{3}$$

$$\underline{x}_{cm} = \frac{0 + a + a}{3} = \frac{2}{3}a$$

$$\underline{y}_{cm} = \frac{0 + 0 + 0}{3} = 0$$

$$\therefore \underline{r}_{cm} = \left(\frac{2}{3}a, 0 \right)$$

$$\underline{\dot{r}}_{cm} = \frac{\underline{\dot{r}}_A + \underline{\dot{r}}_B + \underline{\dot{r}}_C}{3}$$

In Cartesian Coordinates.

$$\underline{r}_A = (x, y)^T \quad \underline{r}_B = (a-x, -y)^T \quad \underline{r}_C = (a, 0)^T$$

$$\therefore \underline{r}_{cm} = \frac{1}{3} \left[\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a-x \\ -y \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix} \right]$$

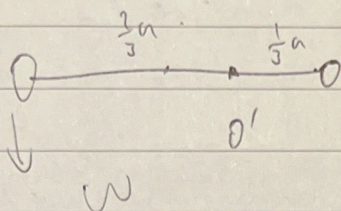
$$= \begin{pmatrix} \frac{2}{3}a \\ 0 \end{pmatrix}$$

$$\underline{\dot{r}}_{cm} = \frac{1}{3} \left[\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} -\dot{x} \\ -\dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \underline{\dot{r}}_{cm} = 0$$

(b)

Just before the collision



$$L_A = m v r_A = m \left(\omega \frac{3a}{2} \right) \left(\frac{3a}{2} \right) = m \left(\omega \frac{3a}{2} \right) \left(\frac{3a}{2} \right) = \frac{9}{4} m a^2 \omega$$

$$L_B = m v r_B = m \left(\omega \frac{a}{2} \right) \left(\frac{a}{2} \right) = \frac{1}{4} m a^2 \omega$$

$$L_C = 0$$

$$\therefore L = L_A + L_B + L_C = \frac{9}{4} m a^2 \omega + \frac{1}{4} m a^2 \omega = \frac{5}{2} m a^2 \omega$$

$\therefore$ No net force $\therefore$ The CM remains stationary after collision

$\therefore$ No net Torque $\therefore$ Angular momentum about O' is conserved before and after the collision.

$$I_{O'} = I_O + m (O'O')^2 = \frac{5}{2} m a^2$$

After collision

$$I = m \left(\frac{3a}{2} \right)^2 + m \left(\frac{a}{2} \right)^2 + m \left(\frac{a}{2} \right)^2 = \frac{4}{3} m a^2 + \frac{1}{3} m a^2 + \frac{1}{3} m a^2 = \frac{2}{3} m a^2$$

$$I' \omega' = L \quad \therefore \omega' = \frac{L}{I'} = \frac{\frac{5}{2} m a^2 \omega}{\frac{2}{3} m a^2} = \frac{3}{4} \omega$$

(c). Before:

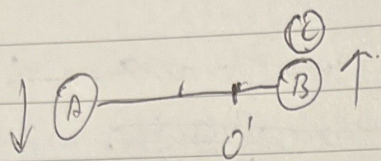
$$T = \frac{1}{2} m \left(\frac{a\omega}{2} \right)^2 + \frac{1}{2} m \left(\frac{a\omega}{2} \right)^2 = m \left(\frac{a\omega}{2} \right)^2 = \frac{m a^2 \omega^2}{4}$$

After:

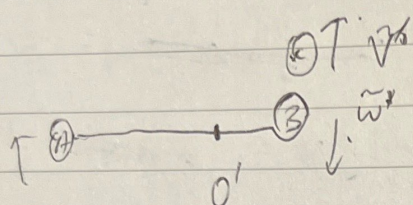
$$T' = \frac{1}{2} I' \omega'^2 = \frac{1}{2} \left(\frac{2}{3} m a^2 \right) \left(\frac{3}{4} \omega \right)^2$$

$$= \frac{1}{3} m a^2 \frac{9}{16} \omega^2 = \frac{3 m a^2 \omega^2}{16}$$

(d) If the collision was totally elastic:



Angular momentum about O' conserved (No net torque).



Kinetic Energy conserved. (elastic?)

Linear Momentum conserved. (No net force).

$$\frac{1}{2} I \omega + m \tilde{v} \left(\frac{1}{3} a \right) = L.$$

$$\therefore \left(\frac{2}{3} m a^2 \right) \tilde{\omega} + m \tilde{v} \left(\frac{1}{3} a \right) = \frac{1}{2} m a^2 \omega \quad (1)$$

$$\frac{1}{2} I \omega^2 + \frac{1}{2} m \tilde{v}^2 = \frac{m a^2 \omega^2}{4}$$

$$\frac{1}{3} m a^2 \tilde{\omega}^2 + \frac{1}{2} m \tilde{v}^2 = \frac{1}{4} m a^2 \omega^2 \quad (2)$$

$$(1) \Rightarrow 4a^2 \tilde{\omega} + 2a \tilde{v} = 3a^2 \omega$$

$$(2) \Rightarrow 4a^2 \tilde{\omega}^2 + 6\tilde{v}^2 = 3a^2 \omega^2$$

(1) $\Rightarrow$

$$2a \tilde{v} = 3a^2 \omega - 4a^2 \tilde{\omega}$$

$$\therefore \tilde{v} = \frac{3}{2} a \omega - 2a \tilde{\omega}$$

$$\therefore \tilde{v}^2 = \frac{9}{4} a^2 \omega^2 - 6a^2 \omega \tilde{\omega} + 4a^2 \tilde{\omega}^2$$

(2) $\Rightarrow$

$$4a^2 \tilde{\omega}^2 + 6 \left(\frac{9}{4} a^2 \omega^2 - 6a^2 \omega \tilde{\omega} + 4a^2 \tilde{\omega}^2 \right) = 3a^2 \omega^2$$

$$\therefore 4a^2 \tilde{\omega}^2 + \frac{27}{2} a^2 \omega^2 - 36a^2 \omega \tilde{\omega} + 24a^2 \tilde{\omega}^2 = 3a^2 \omega^2$$

$$28a^2 \tilde{\omega}^2 - 36a^2 \omega \tilde{\omega} + \frac{21}{2} a^2 \omega^2 = 0$$

$$\therefore 56a^2 \tilde{\omega}^2 - 72a^2 \omega \tilde{\omega} + 21a^2 \omega^2 = 0$$

$$I = \frac{5}{9} m a^2$$

$$\vec{L} = \vec{I} \vec{\omega} + m \vec{r} \times (\frac{1}{3} \vec{a} \vec{\omega}) = 2$$

$$(\frac{5}{9} m a^2) \vec{\omega} + \frac{1}{3} m \vec{r} \times \vec{a} = \frac{1}{2} m a^2 \vec{\omega}$$

$$\vec{r} = \frac{2}{3} \vec{a}$$

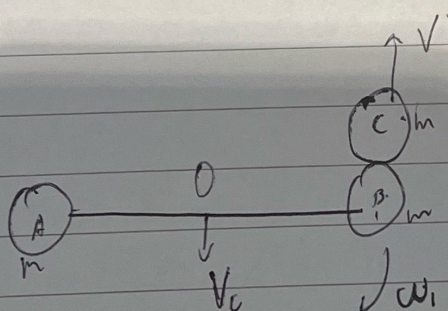
$$10 a^2 \vec{\omega} + 6 a \vec{r} = 9 a^2 \vec{\omega}$$

$$\frac{1}{3} m a^2 (\frac{5}{9} m a^2) \vec{\omega} + \frac{1}{2} m \vec{r} \times \vec{a} = \frac{1}{4} m a^2 \vec{\omega}$$

$$I_c = m (\frac{a}{2})^2 \times 2 = \frac{1}{2} m a^2$$

conservation of linear momentum

conservation of kinetic energy



$$mV = 2mV_c$$

$$\frac{1}{2} (2m) V_c^2 + \frac{1}{2} I_c \omega^2 + \frac{1}{2} m V^2 = \frac{1}{2} I_c \omega^2 \quad (2)$$

$$I_c \omega = I_c \omega + m V \frac{a}{2} + I_c \omega \quad (3)$$

conservation of angular momentum with respect to point O.

$$I_c = \frac{1}{2} m a^2$$

$$(1) \Rightarrow V = 2V_c$$

$$(2) \quad m V_c^2 + \frac{1}{4} m a^2 \omega^2 + \frac{1}{2} m 4V_c^2 = \frac{1}{4} m a^2 \omega^2$$

$$3V_c^2 = \frac{m a^2}{4} (\omega + \omega_1)(\omega - \omega_1) \quad (3)$$

$$(3) \quad \frac{m a^2}{2} = m V_c a = \frac{1}{2} m a^2 (\omega + \omega_1)$$

$$(4) \quad 2V_c = a(\omega + \omega_1) \quad (4)$$

$$(4) \rightarrow (3) \quad 3V_c^2 = \frac{a}{2} V_c (\omega + \omega_1) \Rightarrow 6V_c = a(\omega + \omega_1)$$

$$\omega - \omega_1 = \frac{2V_c}{a} \Rightarrow \frac{6V_c}{a} = 2\omega \Rightarrow V_c = \frac{2a\omega}{8} = \frac{1}{4} a\omega$$

$$\omega + \omega_1 = \frac{6V_c}{a} \Rightarrow \omega_1 = \omega - \frac{2V_c}{a} = \omega - \frac{2}{a} (\frac{1}{4} a\omega)$$

$$= \omega - \frac{1}{2} \omega = \frac{1}{2} \omega$$

$$V = 2V_c = \frac{1}{2} a\omega$$