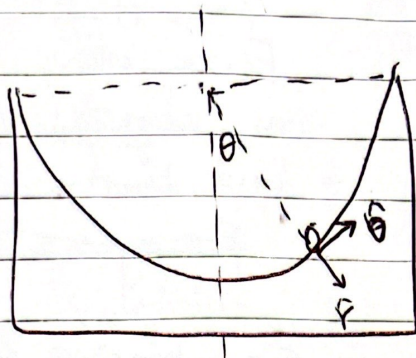


Mechanics 4 Ziyang Li



(a) ~~$\vec{r} = r\hat{r}$~~ $\vec{r} = r\hat{r}$

then velocity

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \boxed{\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$\therefore \frac{d}{dt}(\dot{r}\hat{r}) = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta}$$

$$\frac{d}{dt}(r\dot{\theta}\hat{\theta}) = \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + -r\dot{\theta}^2\hat{r}$$

$$\therefore \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

For this particle

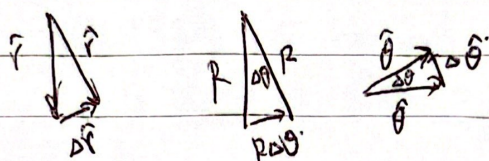
$$\dot{r} = 0, \ddot{r} = 0$$

$$\therefore \vec{a} = r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

$$\therefore v = r\dot{\theta} \therefore \frac{v^2}{r} = \frac{r^2\dot{\theta}^2}{r} = r\dot{\theta}^2$$

$$\therefore \vec{a} = r\ddot{\theta}\hat{\theta} - \frac{v^2}{r}\hat{r}$$

$$\therefore \vec{a} = r\ddot{\theta}\hat{\theta} - \frac{v^2}{r}\hat{r}$$



$$\left|\frac{\Delta\hat{r}}{\Delta\theta}\right| = \left|\frac{|\Delta\hat{r}|}{\Delta\theta}\right| \therefore |\dot{\hat{r}}| = |\dot{\theta}\hat{\theta}|$$

$$\therefore \left|\frac{d\hat{r}}{dt}\right| = |\dot{\theta}\hat{\theta}|$$

$$\therefore \left|\frac{d\hat{r}}{dt}\right| = |\dot{\theta}\hat{\theta}| = \dot{\theta}$$

$$\therefore \hat{r}^2 = 1 \therefore \frac{d\hat{r}^2}{dt} = 2\hat{r} \cdot \frac{d\hat{r}}{dt} = 0$$

$$\therefore \frac{d\hat{r}}{dt} \perp \hat{r} \therefore \frac{d\hat{r}}{dt} \parallel \hat{\theta}$$

$$\therefore \frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

$$\left|\frac{\Delta\hat{\theta}}{\Delta\theta}\right| = |\dot{\theta}|$$

$$\therefore \left|\frac{d\hat{\theta}}{dt}\right| = |\dot{\theta}|$$

$$\therefore \left|\frac{d\hat{\theta}}{dt}\right| = |\dot{\theta}| = \dot{\theta}$$

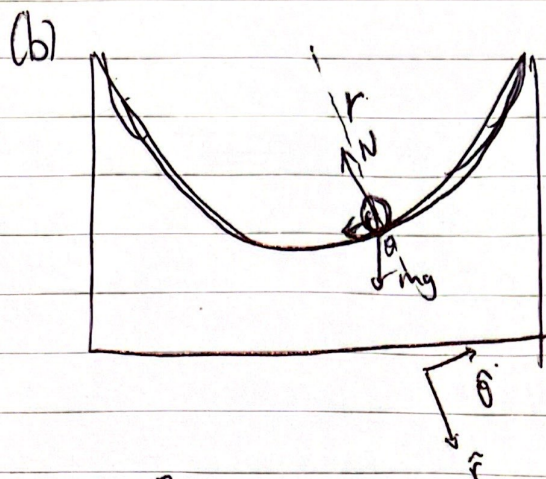
$$\therefore \hat{\theta}^2 = 1$$

$$\therefore \frac{d\hat{\theta}^2}{dt} = 2\hat{\theta} \cdot \frac{d\hat{\theta}}{dt} = 0$$

$$\therefore \frac{d\hat{\theta}}{dt} \perp \hat{\theta} \therefore \frac{d\hat{\theta}}{dt} \parallel -\hat{r}$$

$\therefore d\hat{\theta}$ points to $-\hat{r}$ direction

$$\therefore \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$



Force along $\hat{\theta}$ direction has magnitude $F_{\theta} = mg \sin \theta$ and direction opposite to $\hat{\theta}$

~~$F_{\theta} = mg \sin \theta$~~

$$\therefore \vec{F}_{\theta} = -mg \sin \theta \hat{\theta}$$

Acceleration in $\hat{\theta}$ direction is $\vec{a}_{\theta} = r \ddot{\theta} \hat{\theta}$

Newton's Second Law: $\vec{F}_{\theta} = m \vec{a}_{\theta}$

$$\therefore -mg \sin \theta \hat{\theta} = r \ddot{\theta} \hat{\theta} \quad \because \theta \text{ is small } \therefore \sin \theta \approx \theta$$

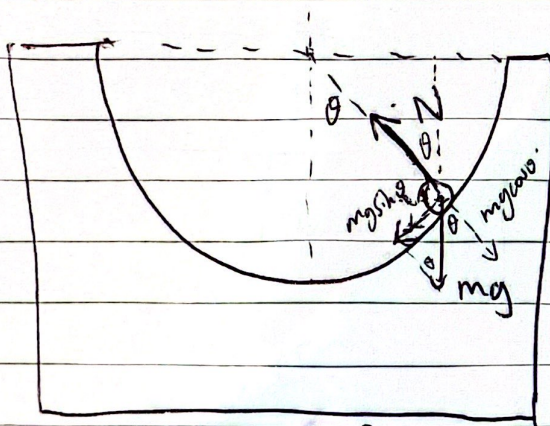
$$\therefore \ddot{\theta} = -\frac{g}{r} \theta \quad \therefore \theta(t) = A \cos(\omega t + \phi)$$

$\therefore$ period is

$$T = 2\pi \sqrt{\frac{r}{g}}$$

$$\omega^2 = \frac{g}{r}$$

(c)



$$N = mg \cos \theta = \frac{v^2}{r} m$$

$$\vec{F} = m \vec{a} = m r \ddot{\theta} \hat{\theta} - m \frac{v^2}{r} \hat{r}$$

Resolve for $\hat{r}$ direction.

$$N - mg \cos \theta = m \frac{v^2}{r}$$

$\therefore$ when ball released from rest,

$$\therefore \frac{1}{2} m v^2 - m g r \cos \theta = 0 \quad \therefore v^2 = 2 g r \cos \theta$$

$$\therefore N = \cancel{mg \cos \theta} + m g \cos \theta + m \frac{v^2}{r} = mg \cos \theta + \frac{m 2 g r \cos \theta}{r}$$

$$= 3 m g \cos \theta$$

$$N(60^\circ) = 3 m g \cos 60^\circ = \boxed{\frac{3}{2} m g}$$

$$2 \quad \frac{d}{dt} (\underline{\dot{r}} \times \underline{r}) = \frac{d\underline{\dot{r}}}{dt} \times \underline{r} + \underline{\dot{r}} \times \frac{d\underline{r}}{dt}$$

$$\cancel{\text{By def}} = \underline{\ddot{r}} \times \underline{r} + \underline{\dot{r}} \times \underline{\dot{r}} \\ = \underline{\ddot{r}} \times \underline{r}$$

By definition $\underline{f(r)} = \underline{\ddot{r}}m/m = \underline{\ddot{r}}$

$\therefore \underline{f(r)}$ is a central force $\therefore \underline{f(r)} \parallel \underline{r}$

$\therefore \underline{f(r)} \times \underline{r} = 0 \therefore \underline{\ddot{r}} \times \underline{r} = 0$

$\therefore \frac{d}{dt} (\underline{\dot{r}} \times \underline{r}) = 0 \therefore \underline{r} \times \underline{\dot{r}}$ is ⁿ constant vector

$\therefore \underline{r} \times \underline{\dot{r}}$ is perpendicular to both $\underline{r}$ and $\underline{\dot{r}}$, and is constant.

$\therefore$ the plane defined by $\underline{r}$ and $\underline{\dot{r}}$ will not change. ($\underline{r} \times (\underline{r} \times \underline{\dot{r}}) = 0$ let $\underline{r} \times \underline{\dot{r}} = \underline{h}$.

then $\underline{r} \cdot \underline{h} = 0$, which is the equation of a plane since $\underline{h}$ is a constant vector

$$\underline{J} = \underline{r} \times m \underline{\dot{r}} \quad |J| = |\underline{r} \times m \underline{\dot{r}}| \\ = |r \hat{r} \times m(\dot{r} \hat{r} + r \dot{\theta} \hat{\theta})|$$

$\therefore |\hat{r} \times \hat{\theta}| = 1 \quad \hat{r} \times \hat{r} = 0 \therefore |J| = mr^2 \dot{\theta}$

$$\frac{d|J|}{dt} = m \frac{d}{dt} (r^2 \dot{\theta}) = r^2 \ddot{\theta} + \dot{\theta} \frac{d}{dt} r^2$$

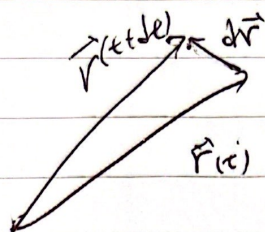
$$= r^2 \ddot{\theta} + 2r\dot{r}\dot{\theta} = r(\ddot{\theta} + 2\dot{r}\dot{\theta}/r)$$

$\therefore$ acceleration in general is $\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$

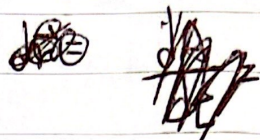
and also $\therefore \underline{f(r)}$ is central force $\therefore$ acceleration

in $\hat{\theta}$ direction is 0.

$\therefore r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \therefore \frac{d|J|}{dt} = 0 \therefore |J|$ is constant



$$dA = \frac{1}{2} |\vec{r} \times d\vec{r}|$$



$$\vec{r} = r\hat{r} + r\hat{\theta}$$

$$d\vec{r} = \hat{r} dr + r d\hat{\theta}$$

$$dA = \frac{1}{2} |\vec{r} \times (\hat{r} dr + r d\hat{\theta})|$$

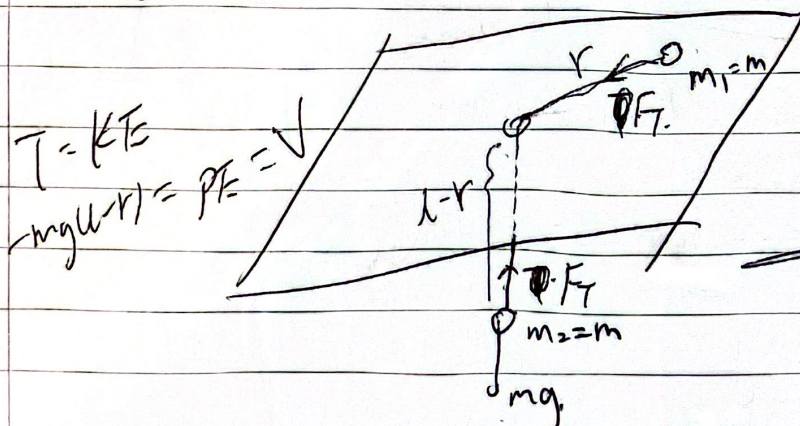
$$= \frac{1}{2} |r\hat{r} \times (\hat{r} dr + r d\hat{\theta})|$$

$$= \frac{1}{2} |r^2 d\hat{\theta}| = \frac{1}{2} r^2 d\theta$$

$$\therefore \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{|J|}{2m}$$

$$\therefore |J| \text{ is constant } \therefore \frac{dA}{dt} \text{ is constant.}$$

(3)



$$T = kx$$

$$mg(l-r) = p_F = V$$



$$\frac{1}{2} m \dot{r}^2 = mg(l-r)$$

$$T - mg(l-r) = A$$

$$T = \frac{1}{2} m \dot{r}^2 + T_{m1} + T_{m2}$$

Assume when $r=l$ $\dot{x}_{m1} = v_i$

then when $r=r$, by conservation of angular momentum,

$$m r v' = m l v_i \therefore v' = \frac{m l v_i}{r}$$

$$T_{m1} = \frac{1}{2} m v'^2 = \frac{1}{2} m \left(\frac{m l v_i}{r} \right)^2 = \frac{m l^2 v_i^2}{2 r^2}$$

$$T_{m2} = \frac{1}{2} m \dot{\theta}^2$$

$$A = \frac{1}{2} m v_i^2$$

$$\therefore T = \frac{1}{2} m \dot{r}^2 + \frac{m l^2 \dot{V}_i^2}{2 r^2} + \frac{1}{2} m \dot{V}^2.$$

$$\therefore T + V = A = \frac{1}{2} m \dot{V}^2.$$

$$\therefore m \ddot{r} + \frac{m l^2 \dot{V}_i^2}{r^3} - \frac{m \dot{V}^2}{r} - mg(l-r) = \frac{1}{2} m \dot{V}_i^2.$$

$$\therefore m \ddot{r} = \left(\frac{1}{2} m \dot{V}_i^2 + mg(l-r) \right) - \frac{m l^2 \dot{V}_i^2}{r^3} - m \dot{V}^2 / r.$$

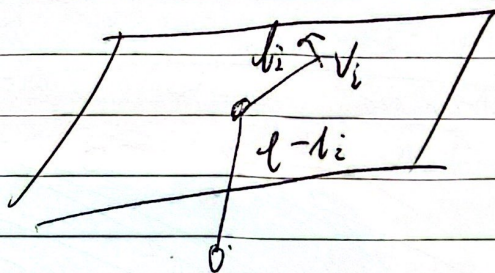
$$\therefore \ddot{r} = \left(\frac{1}{2} \dot{V}_i^2 + gl \right) - \frac{l^2 \dot{V}_i^2}{r^3} - \dot{V}^2 / r.$$

$$\ddot{r} = \frac{1}{2} \dot{V}_i^2 + gl - \frac{l^2 \dot{V}_i^2}{r^3} - \frac{\dot{V}^2}{r}$$

$$\ddot{r} = \frac{1}{2} \dot{V}_i^2 + gl - \frac{l^2 \dot{V}_i^2}{r^3} - \frac{\dot{V}^2}{r}$$

Taking the table to be the level of $V=0$.

Initial condition:



$$E = \frac{1}{2} m \dot{V}_i^2 - mg(l-l_i)$$

$$J = m l_i \dot{V}_i.$$

let V represent transverse ~~velocity~~ component of velocity

$$\Delta J = 0 \Rightarrow m r V = m l_i \dot{V}_i \Rightarrow V = \frac{l_i \dot{V}_i}{r}$$

$$T = \underbrace{\frac{1}{2} m \dot{r}^2}_{m_2} + \underbrace{\left(\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m V^2 \right)}_{m_1} = m \dot{r}^2 + \frac{m l_i^2 \dot{V}_i^2}{2 r^2}$$

$$V = -mg(l-r)$$

$$T + V = E$$

$$\therefore mr'^2 + \frac{m l_i^2 v_i^2}{2r^2} - mg(l-r) = \frac{1}{2} m v_i^2 - mg(l-l_i)$$

$$\therefore r'^2 = \left(\frac{1}{2} m v_i^2 + g l_i \right) - \frac{l_i^2 v_i^2}{2r^2} - gr$$

where $A = \frac{1}{2} m v_i^2 + g l_i$, $B = \frac{l_i^2 v_i^2}{2}$ are constants

When $v_i = v_0$, $l_i = \frac{l}{2}$

~~$$A = \frac{1}{2} m v_0^2 + \frac{1}{2} g l$$~~

$$A = \frac{1}{2} v_0^2 + \frac{1}{2} g l = \frac{v_0^2 + g l}{2}$$

$$B = \frac{(l/2)^2 v_0^2}{2} = \frac{l^2 v_0^2}{8}$$

(b)

Circular motion means $r' = 0$

$$\therefore A - \frac{B}{r^2} - gr = 0 \quad \text{when } r = \frac{l}{2}$$

$$\therefore \frac{v_0^2 + g l}{2} - \frac{v_0^2}{8} - g \frac{l}{2} = 0$$

for ball 2 $F_T = mg$

for ball ①

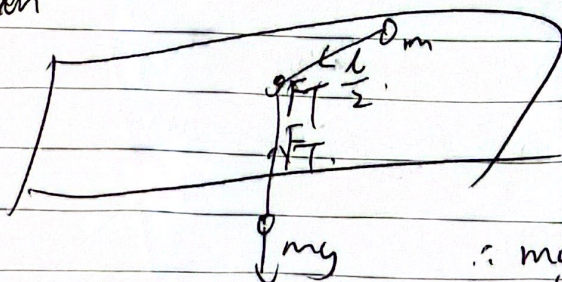
$$F_T = \frac{v_0^2}{r} m = \frac{2 v_0^2}{l} m$$

$$\therefore mg = \frac{2 v_0^2}{l} m$$

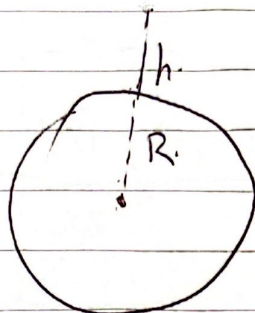
$$\therefore \frac{v_0^2}{g l} = \frac{1}{2}$$

~~Analyse the force~~

Analyse forces



4 (a)



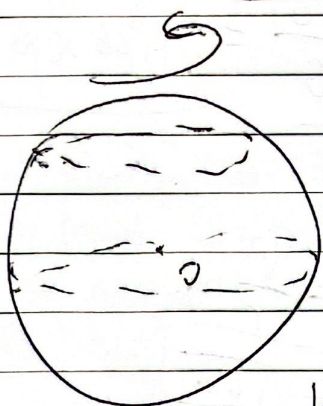
$$\frac{GMm}{(R+h)^2} = \omega^2 (R+h) m.$$

$$\therefore R+h = \sqrt[3]{\frac{GM}{\omega^2}}, \text{ where } M = 5.97 \times 10^{24} \text{ kg}$$

$$\omega = \frac{2\pi}{24 \times 3600 \text{ s}}, \quad R = 6371 \times 10^3 \text{ m.}$$

$$\therefore h = \sqrt[3]{\frac{GM}{\omega^2}} - R = \sqrt[3]{\frac{6.67 \times 10^{-11} (5.97 \times 10^{24})}{(7.27 \times 10^{-5})^2}} - 6371 \times 10^3$$

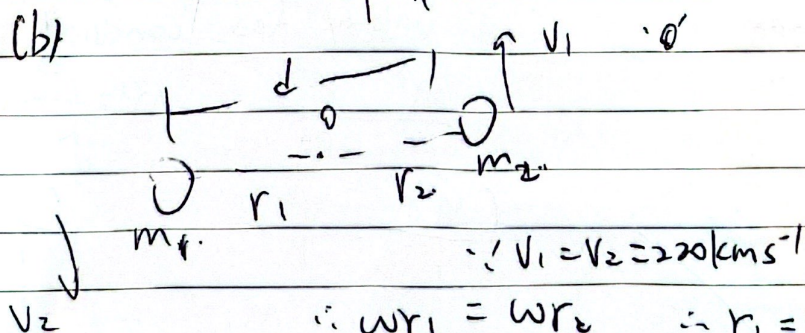
$$= \boxed{35856 \text{ km}}$$



The plane of motion of the satellite must contain the centre of Earth O , and the plane of motion of the city must be a plane normal to the axis of rotation of earth.

Hence the only possibility in which both conditions are satisfied is when the city is on the Equator and the satellite orbits above the equator.

(b)



$$\therefore v_1 = v_2 = 220 \text{ km/s}$$

$$\therefore \omega r_1 = \omega r_2 \quad \therefore r_1 = r_2.$$

$$\therefore O \text{ is centre of mass} \quad \therefore m_1 r_1 = m_2 r_2.$$

$$\therefore \underline{m_1 = m_2.}$$

let $r_1 = r_2 = r$,

then $\omega r = v \quad \therefore \frac{2\pi}{T} r = v$

$$\therefore r = \frac{vT}{2\pi} = \frac{220 \times 10^3 \text{ m/s} \times 14.4 \times 24 \times 3600 \text{ s}}{2\pi}$$

$$= 4.36 \times 10^{10} \text{ m.}$$

$$\therefore d = 2r = 8.7 \times 10^{10} \text{ m}$$

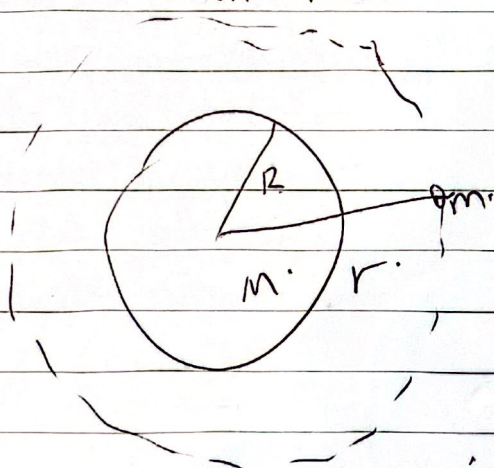
$$F_g = \frac{Gm^2}{d^2} = \frac{v^2}{r} m.$$

~~$$\therefore m = \frac{v^2 d^2}{rG} = \frac{v^2}{rG} m$$~~

$$\therefore m = \frac{v^2 d}{G} = 1.26 \times 10^{34} \text{ kg}$$

5

when $r > R$



$$\frac{GMm}{r^2} = \frac{v^2}{r} m.$$

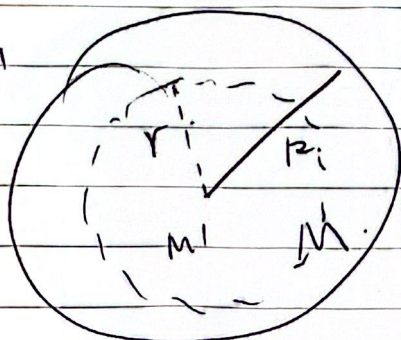
$$\therefore V = \sqrt{\frac{GM}{r}}$$

when $r < R$

$\therefore$ Galaxy has constant density

$$\therefore \frac{M'}{M} = \frac{r^3}{R^3} \quad \therefore M' = \frac{r^3}{R^3} M.$$

inner sphere
has
mass M'

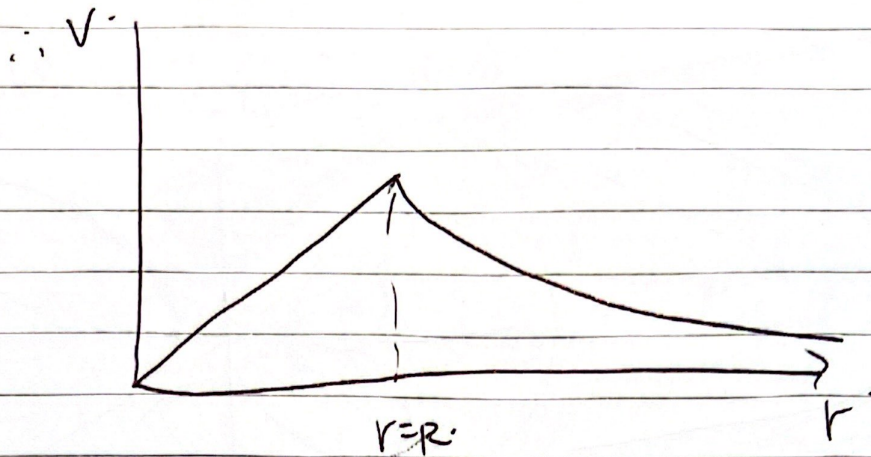


$$\frac{GM'm}{r^2} = \frac{v^2}{r} m \quad \therefore \frac{GM'}{r} = v^2.$$

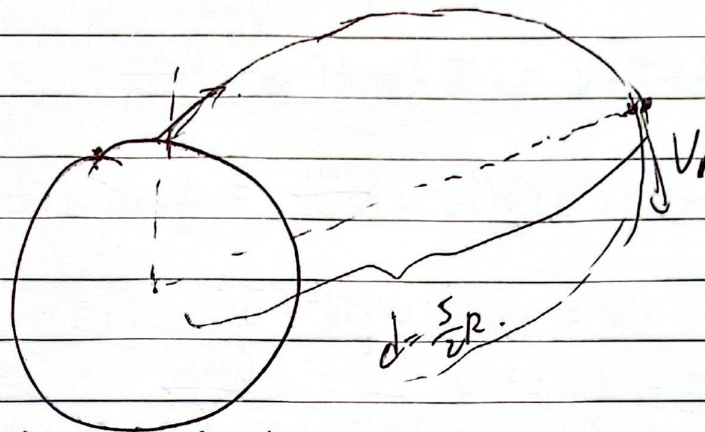
$$\therefore \frac{Gr^3 M / R^3}{r} = v^2$$

$$\therefore \frac{GM}{R^3} r^2 = v^2$$

$$\therefore V = r \sqrt{\frac{GM}{R^3}}$$



6.



Conservation of angular momentum

At launch $J = m R V_0 \sin 30^\circ = \frac{1}{2} m R V_0$

At Apogee : $J = \frac{1}{2} m R V_0 = m \left(\frac{5}{2} R \right) V_1$

$\therefore V_1 = \frac{1}{5} V_0$

Conservation of ~~Energy~~ energy

At launch : $E = \frac{1}{2} m V_0^2 - \frac{G M m}{R}$

At Apogee : $E = \frac{1}{2} m V_1^2 - \frac{G M m}{\frac{5}{2} R} = \frac{1}{2} m V_1^2 - \frac{2 G M m}{5 R}$

$\therefore V_0^2 - \frac{2 G M}{R} = V_1^2 - \frac{4 G M}{5 R} = \frac{1}{25} V_0^2 - \frac{4 G M}{5 R}$

$\therefore \frac{24}{25} V_0^2 = \left(2 - \frac{4}{5} \right) \frac{G M}{R} = \frac{6}{5} \frac{G M}{R}$

$\therefore V_0^2 = \frac{\frac{25}{24} \times 6}{4} \frac{G M}{R} = \boxed{\frac{5 G M}{4 R}}$

$\frac{24}{25} V_0^2 = \frac{6 G M}{5 R}$
 $V_0^2 = \frac{25 \times 6 G M}{24 \times 5 R}$
 $V_0^2 = \frac{5 G M}{4 R}$

7 (a)



$$(b) \quad E = T + V = \frac{1}{2} m |\dot{\mathbf{r}}|^2 - \frac{Gmm}{|\mathbf{r}|}$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{Gmm}{r} = \frac{1}{2} m \dot{r}^2 + \frac{m^2 r^4 \dot{\theta}^2}{2mr^2} - \frac{Gmm}{r}$$

$$\text{or } J = mr^2 \dot{\theta} \quad \therefore m^2 r^4 \dot{\theta}^2 = J^2$$

$$\therefore E = \frac{1}{2} m \dot{r}^2 + \frac{J^2}{2mr^2} - \frac{Gmm}{r} \quad \text{or } J = mr^2 \dot{\theta} \quad \alpha = Gmm$$

(c)

At closest approach $\dot{r} = 0$

$$\therefore E = \frac{J^2}{2mr^2} - \frac{Gmm}{r}$$

$$E = \frac{1}{2} m v_0^2$$

$$J = m v_0 b$$

$$\therefore \frac{1}{2} m v_0^2 = \frac{m^2 v_0^2 b^2}{2mr^2} - \frac{Gmm}{r}$$

~~or~~

$$v_0^2 = \frac{v_0^2 b^2}{r^2} - \frac{2GM}{r}$$

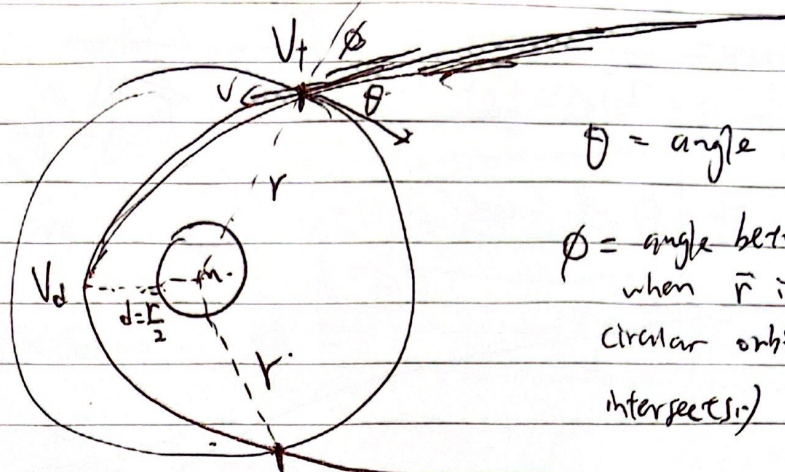
$$\therefore v_0^2 r^2 + 2GM r - v_0^2 b^2 = 0$$

$$\therefore r_{\min} = \frac{-2GM \pm \sqrt{4G^2 M^2 - 4v_0^4 b^2}}{2v_0^2}$$

$\therefore r_{\min} > 0$

$$\therefore r_{\min} = \frac{1}{v_0^2} \left\{ \sqrt{G^2 M^2 + v_0^4 b^2} - GM \right\}$$

(d)



θ = angle between orbits

ϕ = angle between $\vec{r}$ and $\vec{V}_{\text{tangent}}$ when $\vec{r}$ is radius of the circular orbit. (when the orbits intersect) $\phi + \theta = 90^\circ$

Conservation of Energy:

$$\frac{1}{2} m V_d^2 - \frac{GMm}{d} = E = \frac{1}{2} m V_t^2 - \frac{GMm}{r}$$

$$\therefore d = \frac{r}{2} \quad \therefore V_d^2 - \frac{2GM}{r} = V_t^2 - \frac{2GM}{r}$$

$$\therefore V_t^2 = V_d^2 - \frac{2GM}{r} \quad (1)$$

Conservation of Angular momentum:

$$m r V_t \sin \phi = m V_d (d) = \frac{1}{2} m V_d r$$

$$V_t \sin \phi = \frac{V_d}{2}$$

$$\therefore \sin \phi = \frac{V_d}{2 V_t} = \frac{V_d}{2 \sqrt{V_d^2 - \frac{2GM}{r}}}$$

$$\therefore \theta + \phi = 90^\circ \quad \therefore \cos \theta =$$

$$\therefore \cos \theta = \frac{V_d}{2 \sqrt{V_d^2 - \frac{2GM}{r}}}$$

$$V_d = 6 \times 10^3 \text{ m/s} \quad G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \quad M = 1.99 \times 10^{30} \text{ kg}$$

$$r = 1.5 \times 10^{11} \text{ m}$$

$$M = 1.99 \times 10^{30} \text{ kg}$$

According to the conditions given - $2 \sqrt{\frac{GM}{r}} = V_d$

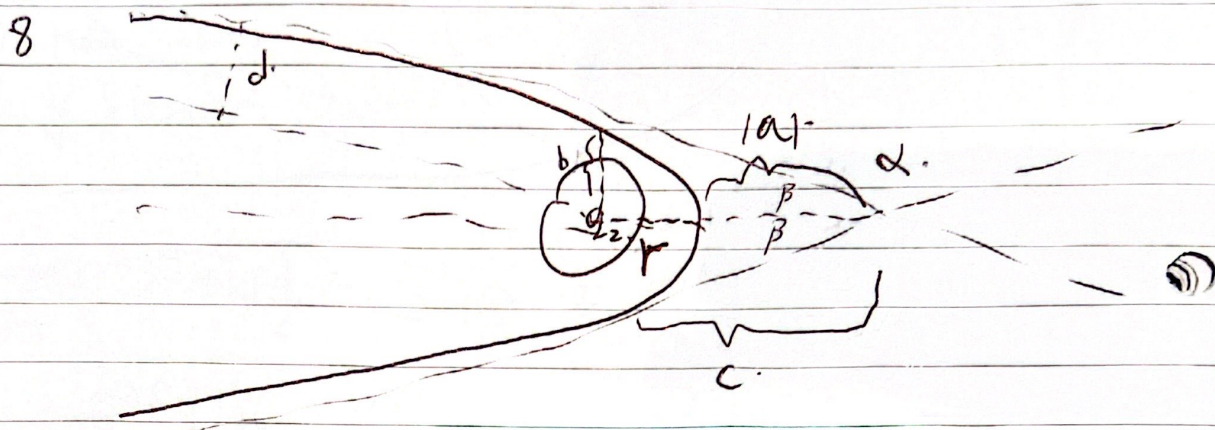
$$\therefore V_d^2 = \frac{4GM}{r}$$

$$\therefore \frac{GM}{r} = \frac{V_d^2}{4}$$

$$\therefore \frac{2GM}{r} = \frac{V_d^2}{2}$$

$$\therefore \cos \theta = \frac{V_d}{2\sqrt{V_d^2 - \frac{1}{2}V_d^2}} = \frac{V_d}{2\sqrt{\frac{1}{2}V_d^2}} = \frac{V_d}{\frac{2}{\sqrt{2}}V_d} = \frac{\sqrt{2}}{2}$$

$$\therefore \boxed{\theta \approx 45^\circ}$$



For an inverse square force $\vec{F}(\vec{r}) = -\frac{C}{r^2} \hat{r}$,
 total energy is $E = -\frac{K}{2a}$, where a is the
 length of semi-major axis of the conic curve.

In the case of a hyperbola, $a < 0 \therefore E > 0$.
 and $E = \frac{K}{2|a|} \therefore |a| = \frac{K}{2E}$

According to the previous problem, the minimum distance
 between the hyperbola (orbit) and focus (g_2), r , is
 given by $E = \frac{J^2}{2mr^2} - \frac{K}{r}$

$$\therefore 2mEr^2 + 2mKr - J^2 = 0 \quad 2mEr^2 + 2mKr - J^2 = 0$$

$$\therefore r = \frac{-K \pm \sqrt{K^2 + 8mEJ^2}}{4mE} \quad r = \frac{-2mK + \sqrt{4m^2K^2 + 8mEJ^2}}{4mE}$$

let the equation of hyperbola in cartesian be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and in polar be $r = \frac{r_0}{1 - \epsilon \cos \theta}$

wh we have $c^2 = a^2 + b^2$ and $\epsilon = \frac{c}{a}$.
 $c = |a| + r$

$$c = |a| + r = \frac{K}{2E} + \frac{-2mK + \sqrt{4m^2K^2 + 8mEJ^2}}{4mE}$$

$$= \frac{\sqrt{4m^2K^2 + 8mEJ^2}}{4mE} = \frac{2mK \sqrt{1 + \frac{2EJ^2}{mK^2}}}{4mE}$$

$$= \frac{K}{2E} \sqrt{1 + \frac{2EJ^2}{mK^2}}$$

$$\therefore \epsilon = \frac{c}{a} = \sqrt{1 + \frac{2EJ^2}{mK^2}}$$

Considering the polar equation $r = \frac{r_0}{1 - \epsilon \cos \theta}$
 when $\theta = \beta$, $r = \infty$ $\therefore 1 - \epsilon \cos \beta = 0$

$$\therefore \cos \beta = \frac{1}{\epsilon}$$

$$\alpha + 2\beta = \pi \quad \therefore \alpha = \pi - 2\beta = 2\left(\frac{\pi}{2} - \beta\right)$$

$$\therefore \frac{\alpha}{2} = \frac{\pi}{2} - \beta \quad \therefore \cot \frac{\alpha}{2} + \tan \beta = \frac{\pi}{2}$$

$$\therefore \cot \frac{\alpha}{2} = \tan \beta$$

$$\therefore \cos \beta = \frac{1}{\epsilon} \quad 1 + \tan^2 \beta = \frac{1}{\cos^2 \beta} = \epsilon^2$$

$$\therefore \tan^2 \beta = \epsilon^2 - 1 \quad \therefore \tan \beta = \sqrt{\epsilon^2 - 1}$$

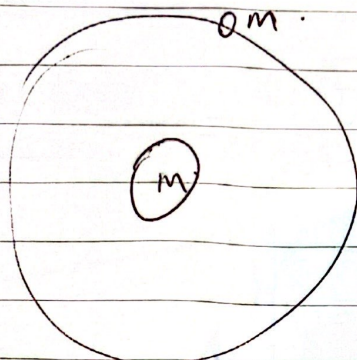
$$\therefore \tan \beta = \sqrt{1 + \frac{2EJ^2}{mK^2} - 1} = \sqrt{\frac{2EJ^2}{mK^2}} = \sqrt{\frac{E}{m}} \frac{J}{K}$$

In this case $F = -\frac{K}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \quad \therefore K = \frac{q_1 q_2}{4\pi\epsilon_0}$

$$J = m v_0 d, \quad E = \frac{1}{2} m v_0^2$$

$$\therefore \cot \frac{\alpha}{2} = \tan \beta = \frac{\sqrt{2E} \frac{J}{k}}{\sqrt{\frac{2E}{m}} \frac{m v_0 d}{(q_1 q_2 / 4\pi \epsilon_0)}} = \boxed{\frac{4\pi \epsilon_0}{2 q_1 q_2} d v_0^2 m} \quad QED$$

(9) (a)



$$\vec{F} = m \vec{a}$$

$$\therefore \frac{GMm}{r_0^2} = \frac{v_0^2}{r_0} m$$

$$\therefore v_0^2 = \frac{GM}{r_0}$$

$$\therefore r_0 = \frac{GM}{v_0^2}$$

$$(b) \quad J_0 = m v_0 r_0 = m v_0 \frac{GM}{v_0^2} = \boxed{\frac{GMm}{v_0}}$$

$$E_0 = T + V = \frac{1}{2} m v_0^2 - \frac{GMm}{r_0}$$

$$= \frac{1}{2} m v_0^2 - \frac{GMm}{GM} v_0^2 = \frac{1}{2} m v_0^2 - m v_0^2 = \boxed{-\frac{1}{2} m v_0^2}$$

(c)

$$E = T + V = \frac{1}{2} m \dot{r}^2 - \frac{GMm}{r} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{GMm}{r}$$

$$= \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{GMm}{r} = \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{(m r^2 \dot{\theta})^2}{2 m r^2} - \frac{GMm}{r}$$

$$= \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{J^2}{2 m r^2} - \frac{GMm}{r}$$

(d) The new orbit is an ellipse.

At $r_{\max}$ and $r_{\min}$ $\frac{dr}{dt} = 0$

$$E = -\frac{GMm}{2r}$$

$$\therefore E = \frac{(LJ_0)^2}{2mr^2} - \frac{GMm}{r} = -\frac{GMm}{2r_0}$$

$$J_0 = \frac{GMm}{v_0} = \frac{GMm}{\frac{J_0 m}{r}}$$

$$J_0 = mv_0 r_0 = m \sqrt{\frac{GM}{r_0}} r_0$$

$$\therefore J_0^2 = \frac{GMm^2}{r_0}$$

$$\therefore \frac{J^2}{2mr^2} = \frac{\alpha^2 GMm^2 r_0}{2mr^2} = \frac{\alpha^2 GMm r_0}{2r^2}$$

$$\therefore -\frac{GMm}{2r_0} = \frac{\alpha^2 GMm r_0}{2r^2} - \frac{GMm}{r}$$

$$\therefore -\frac{1}{2r_0} = \frac{\alpha^2 r_0}{2r^2} - \frac{1}{r}$$

$$\therefore -1 = \frac{\alpha^2 r_0^2}{r^2} - \frac{2r_0}{r}$$

$$\therefore r^2 - 2r_0 r + \alpha^2 r_0^2 = 0$$

$$\therefore r = \frac{2r_0 \pm \sqrt{4r_0^2 - 4\alpha^2 r_0^2}}{2} = \frac{2r_0 \pm 2r_0 \sqrt{1-\alpha^2}}{2}$$

$$= r_0 (1 \pm \sqrt{1-\alpha^2})$$

$$\therefore r_{\max} = r_0 (1 + \sqrt{1-\alpha^2})$$

$$r_{\min} = r_0 (1 - \sqrt{1-\alpha^2})$$

$$\text{If } \alpha = \frac{12}{13}, \sqrt{1-\alpha^2} = \frac{5}{13}$$

$$\therefore r_{\max} = (1 + \frac{5}{13}) r_0 = \frac{18}{13} r_0$$

$$r_{\min} = (1 - \frac{5}{13}) r_0 = \frac{8}{13} r_0$$

$$\frac{mg}{\pi}$$



$$F dx =$$

BR

e.

~~The new energy is~~
 ~~$E' = \beta E$ where $\beta = 1.01$~~

New kinetic energy is $T' = \beta T$, $\beta = 1.01$

$$\therefore T = \frac{GMm}{2r_0} \therefore T' = \frac{\beta GMm}{2r_0}$$

$$\therefore \text{Total energy } E' = \frac{\beta GMm}{2r_0} - \frac{GMm}{r_0} = \left(\frac{\beta}{2} - 1\right) \frac{GMm}{r_0}$$

$$\text{The equation } \frac{1}{2}mr\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{GMm}{r} = E$$

Can be written as ~~$\frac{1}{2}mr\dot{r}^2 + U(r)$~~

$\frac{1}{2}mr\dot{r}^2 + U(r) = E$ where $U(r)$ is the effective

potential energy such that $U(r) = \frac{J^2}{2mr^2} - \frac{GMm}{r} = \frac{J^2}{2mr^2} - U(r)$

$\therefore$ Angular momentum is unchanged by the impulse

$$\therefore \text{In this case } U(r) = \frac{J^2}{2mr^2} - \frac{GMm}{r} = \frac{GMmr_0}{2r^2} - \frac{GMm}{r}$$

$$= GMm \left(\frac{r_0}{2r^2} - \frac{1}{r} \right)$$

At maximum or minimum r , $U(r) = E$

$$\therefore \left(\frac{\beta}{2} - 1\right) \frac{GMm}{r} = GMm \left(\frac{r_0}{2r^2} - \frac{1}{r} \right)$$

$$\therefore \left(\frac{\beta}{2} - 1\right) \frac{1}{r_0} = \frac{r_0}{2r^2} - \frac{1}{r}$$

$$\therefore \frac{(\beta - 2)}{r_0} r^2 = r_0 - 2r$$

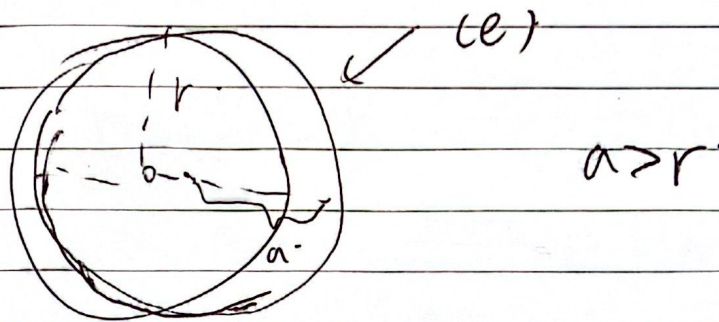
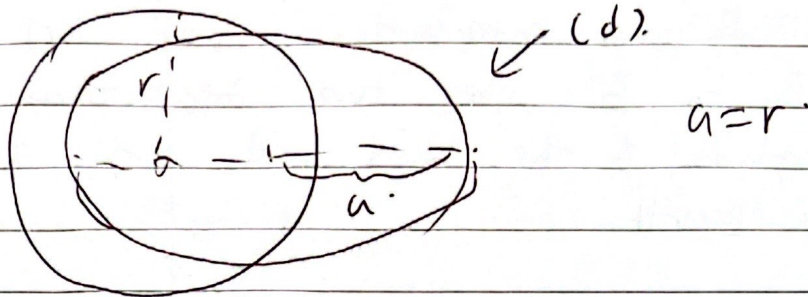
$$\therefore \frac{(2 - \beta)}{r_0} r^2 - 2r + r_0 = 0$$

$$\therefore (2 - \beta)r^2 - 2r_0r + r_0^2 = 0$$

$$\therefore r = \frac{2r_0 \pm \sqrt{4r_0^2 - 4(2 - \beta)r_0^2}}{4 - 2\beta}$$

$$\Delta r = r_{\max} - r_{\min} = (2) \frac{2r_0 \sqrt{1-2\beta}}{2(2-\beta)} = \left(\frac{2\sqrt{1-\beta}}{2-\beta} \right) r_0$$

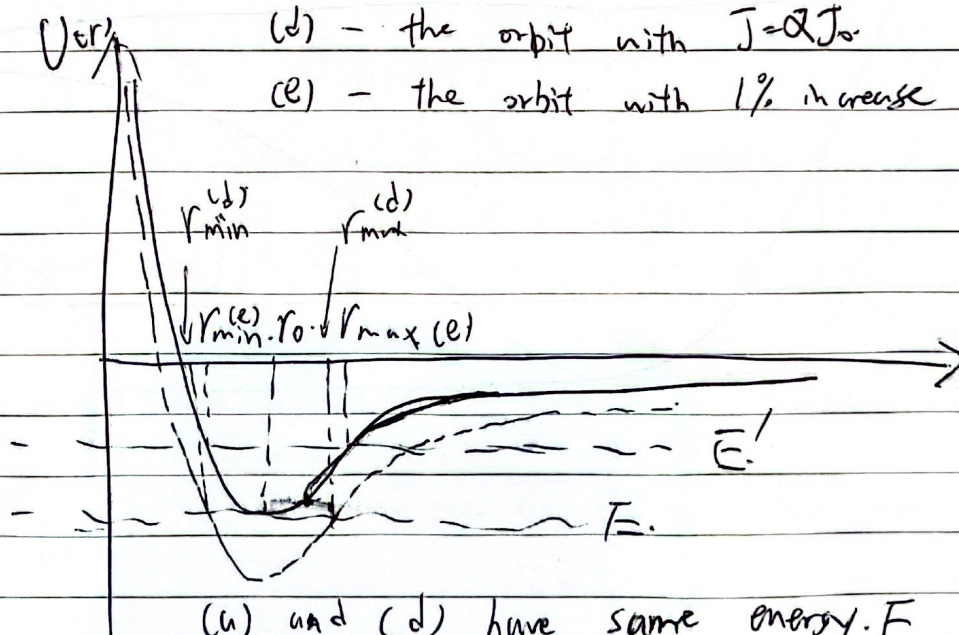
$$\therefore \beta = 1.01 \quad \therefore \Delta r = \underline{0.20 r_0}$$



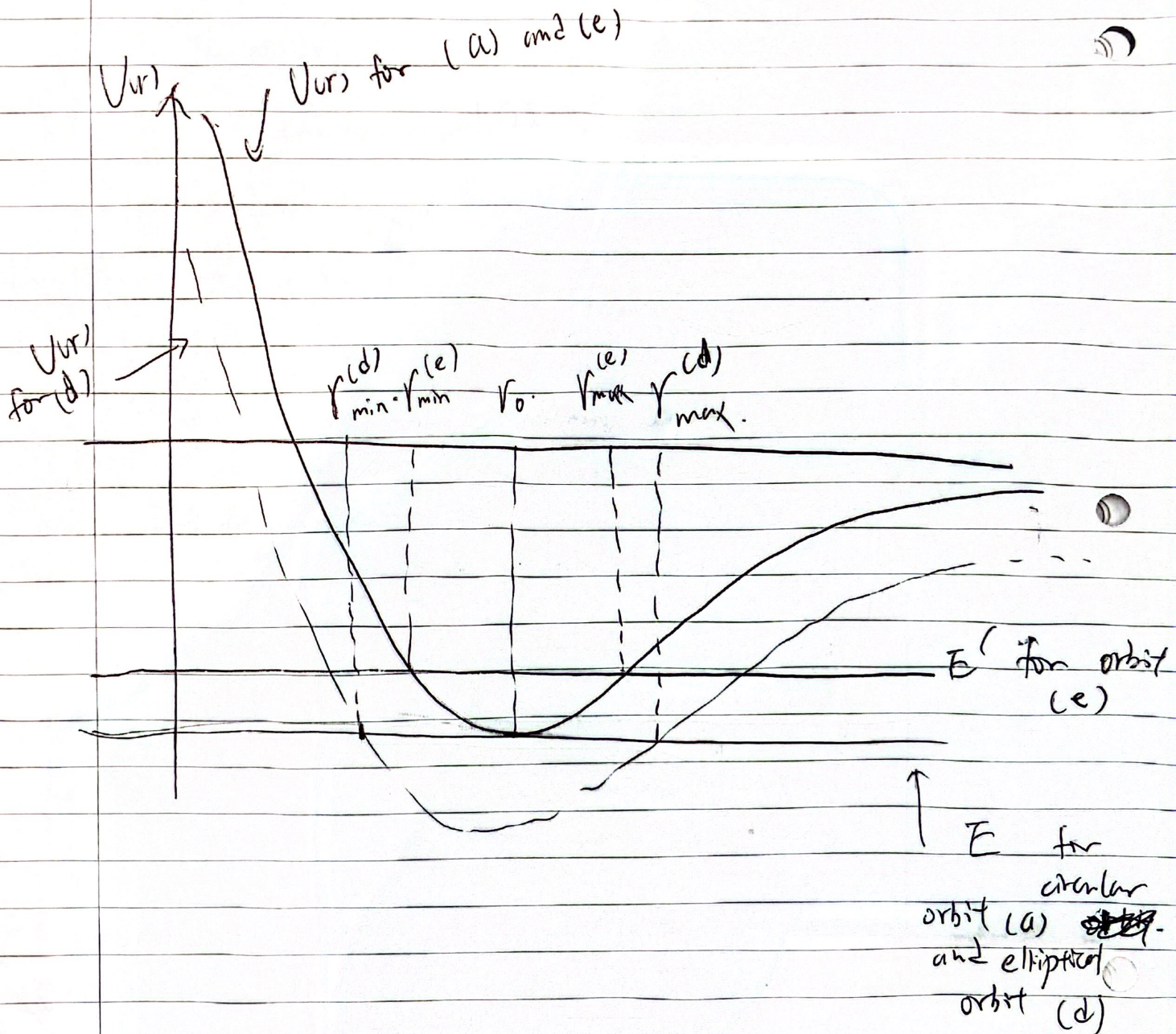
(a) - the original circular orbit

(d) - the orbit with $J = 2J_0$

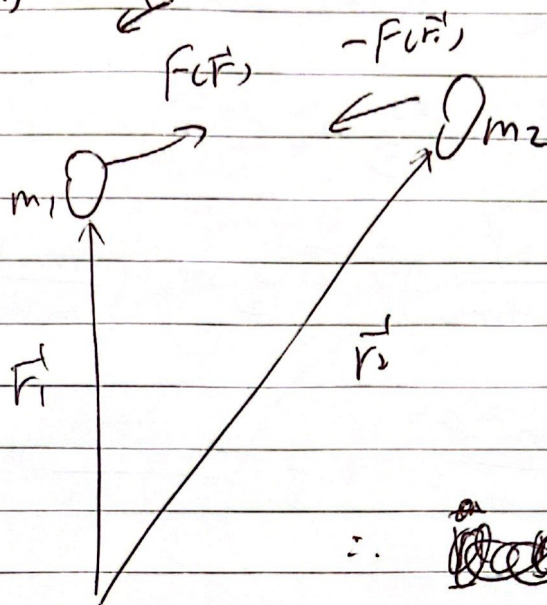
(e) - the orbit with 1% increase in KE.



(a) and (e) have same angular momentum and thus same effective potential energy function. But (e) has higher energy than (a). Since (a) represents the ~~lower~~ lowest point of $V(r)$ and its energy E has only one intersection with $V(r)$ curve, rising E to E' gives two intersections which corresponds to the max and min distances from the Earth.



(10) (a) $\hat{r} = \frac{\underline{r}_2 - \underline{r}_1}{|\underline{r}_2 - \underline{r}_1|}$



$$\underline{F(r)} = m_1 \underline{\ddot{r}_1}$$

$$\underline{F(r)} = m_1 \underline{\ddot{r}_1}$$

$$-\underline{F(r)} = m_2 \underline{\ddot{r}_2}$$

$$\therefore \underline{\ddot{r}_1} = \frac{\underline{F(r)}}{m_1}$$

$$\underline{\ddot{r}_2} = \frac{-\underline{F(r)}}{m_2}$$

$$\therefore \underline{\ddot{r}_1} - \underline{\ddot{r}_2} = \underline{\ddot{r}} = \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \underline{F(r)}$$

$$\therefore \underline{F(r)} = \frac{m_1 m_2}{m_1 + m_2} \underline{\ddot{r}} = \mu \underline{\ddot{r}}$$

$$\therefore \underline{F(r)} = -\frac{GMm_1m_2}{r^2} \hat{r}$$

$$\therefore -\frac{GMm_1m_2}{r^2} \hat{r} = \mu \underline{\ddot{r}}$$

(b) Velocity of centre of mass

$$\underline{V_{cm}} = \frac{m_1 \underline{\dot{r}_1} + m_2 \underline{\dot{r}_2}}{m_1 + m_2}$$

Velocities of m_1, m_2 in CM Frame:

$$\underline{\dot{r}_1'} = \underline{\dot{r}_1} - \underline{V_{cm}} = \frac{m_2 (\underline{\dot{r}_1} - \underline{\dot{r}_2})}{m_1 + m_2} = \frac{m_2 \underline{\dot{r}}}{m_1 + m_2}$$

$$\underline{\dot{r}_2'} = \underline{\dot{r}_2} - \underline{V_{cm}} = \frac{m_1 (\underline{\dot{r}_2} - \underline{\dot{r}_1})}{m_1 + m_2} = \frac{-m_1 \underline{\dot{r}}}{m_1 + m_2}$$

In an ~~Frame~~ Frame, Kinetic Energy T' is

$$T' = \frac{1}{2} m_1 \dot{r}_1'^2 + \frac{1}{2} m_2 \dot{r}_2'^2.$$

$$= \frac{1}{2} m_1 \left(\frac{m_2 \dot{r}}{m_1 + m_2} \right)^2 + \frac{1}{2} m_2 \left(\frac{-m_1 \dot{r}}{m_1 + m_2} \right)^2.$$

$$= \frac{1}{2} \frac{m_1 m_2^2 \dot{r}^2}{(m_1 + m_2)^2} + \frac{1}{2} \frac{m_2 m_1^2 \dot{r}^2}{(m_1 + m_2)^2}$$

$$= \frac{1}{2} \frac{m_1 m_2 (m_1 + m_2)}{(m_1 + m_2)(m_1 + m_2)} \dot{r}^2 = \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) \dot{r}^2.$$

$$= \cancel{\frac{1}{2} \mu \dot{r}^2} \quad \underline{\underline{\frac{1}{2} \mu \dot{r}^2.}}$$

(C) Thus a ~~particle~~ particle of mass μ that experiences the ~~same~~ same net force as the internal force between m_1 and m_2 has the same motion as the relative motion between m_1 and m_2 .

Total energy of system $E = T + V$

$$T = \frac{1}{2} \mu \dot{r}^2 \quad V = - \int_{\infty}^r f(r) \hat{r} \cdot d\hat{r}$$

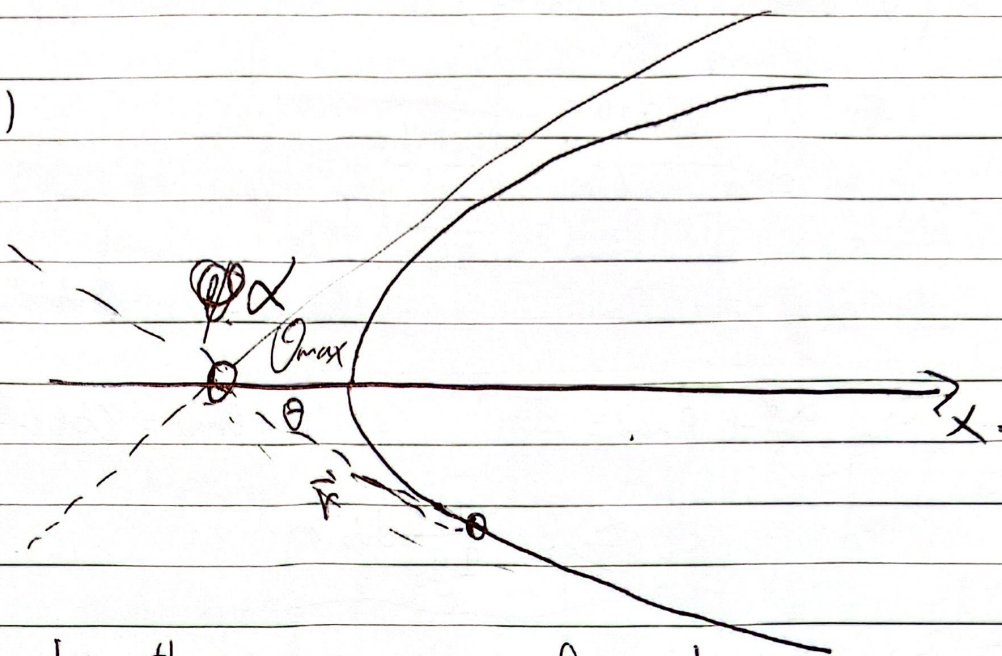
$$\therefore E = \frac{1}{2} \mu \dot{r}^2 - \int_{\infty}^r f(r) \hat{r} \cdot d\hat{r}$$

In the case $f(r) \hat{r} = -\frac{Gm_1 m_2}{r^2} \hat{r}$

$$\boxed{E = \frac{1}{2} \mu \dot{r}^2 - \frac{Gm_1 m_2}{r}} \quad , \quad \boxed{U(r) = \frac{J^2}{2\mu r^2} - \frac{Gm_1 m_2}{r}}$$

where $\underline{J} = \underline{\mu r} \times \dot{\underline{r}}$ is the angular momentum

(11)



Let the inverse square force be

$$\vec{F} = \frac{C}{r^2} \hat{r}, \text{ its magnitude } F = \frac{C}{r^2}$$

Resolve in X direction $m \frac{d^2x}{dt^2} = F_x$

$$\therefore m \frac{d^2x}{dt^2} = \frac{C}{r^2} \cos \theta$$

$$\therefore \frac{d^2x}{dt^2} = \frac{C}{mr^2} \cos \theta = \frac{C}{mr^2 \dot{\theta}} \cos \theta (\dot{\theta})$$

Integrate both sides with respect to time and taking into account of the fact that $mr^2\dot{\theta} = J$, where J is the magnitude of angular momentum of the incident particle.

$$\int \frac{dx}{dt^2} dt = \int \frac{C}{J} \cos \theta \frac{d\theta}{dt} dt.$$

$$\frac{dx}{dt} = \frac{C}{J} \sin \theta + A, \text{ where } A \text{ is a constant.}$$

when $\theta = 0$, ~~the~~ particle is at closest approach.

$$\therefore \frac{dx}{dt} = 0 \quad \therefore A + 0 = 0 \quad \therefore A = 0$$

$$\therefore \frac{dx}{dt} = \frac{C}{J} \sin \theta$$

$$\text{At } \theta = \theta_{\max}, \frac{dx}{dt} = v_0 \cos \theta, J = m v_0 d, C = \frac{q_1 q_2}{4\pi\epsilon_0}$$

$$\therefore v_0 \cos \theta = \frac{q_1 q_2}{4\pi\epsilon_0 m v_0 d} \sin \theta$$

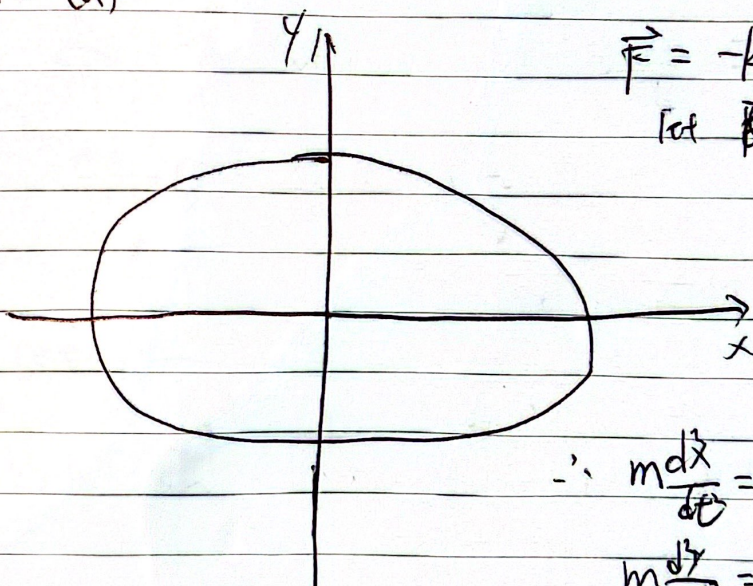
$$\therefore \tan \theta_{\max} = \frac{4\pi\epsilon_0}{q_1 q_2} v_0^2 d m$$

$$\therefore \alpha = \pi - 2\theta_{\max} \therefore 2\theta_{\max} = \pi - \alpha \therefore \theta_{\max} = \frac{\pi}{2} - \frac{\alpha}{2}$$

$$\therefore \frac{\alpha}{2} + \theta_{\max} = \frac{\pi}{2} \therefore \tan \theta_{\max} = \cot\left(\frac{\alpha}{2}\right)$$

$$\therefore \cot\left(\frac{\alpha}{2}\right) = \frac{4\pi\epsilon_0}{q_1 q_2} v_0^2 d m$$

12. (a)



$$\vec{F} = -kr\vec{r}$$

$$\text{Let } \vec{F} = (F_x, F_y)$$

$$\vec{r} = (x, y)$$

$$\text{then } F_x = -kx$$

$$F_y = -ky$$

$$\therefore m \frac{dx}{dt} = -kx \quad (1)$$

$$m \frac{dy}{dt} = -ky \quad (2)$$

$$(1) \rightarrow x(t) = A_1 \cos \omega t + B_1 \sin \omega t$$

$$(2) \rightarrow y(t) = A_2 \cos \omega t + B_2 \sin \omega t$$

$$\left(\omega^2 = \frac{k}{m} \right)$$

$$\dot{x} = -\omega A_1 \sin \omega t + \omega B_1 \cos \omega t$$

$$\dot{y} = -\omega A_2 \sin \omega t + \omega B_2 \cos \omega t$$

$$\therefore \text{At } t=0, \vec{r}_0 = (x_0, 0) \therefore x_0 = A_1, 0 = A_2$$

$$\vec{v}_0 = (0, u_0) \therefore 0 = B_1, u_0 = \omega B_2$$

$$\therefore X(t) = X_0 \cos \omega t, \quad Y(t) = \frac{U_0}{\omega} \sin \omega t. \quad (3)$$

Equation (3) represents a parametric equation of an ellipse centered at origin.

Let major axis be a , minor axis be b .

$$\text{then } a = X_0, \quad b = \frac{U_0}{\omega} \quad \text{if } X_0 > \frac{U_0}{\omega}$$

$$a = \frac{U_0}{\omega}, \quad b = X_0 \quad \text{if } X_0 < \frac{U_0}{\omega}.$$

(b)

$$\vec{F} = -k\vec{r} \quad \therefore V(r) = -\int_0^r F(r') \cdot dr'$$

$$\therefore \text{potential energy } V(r) = -\int_0^r -kr' dr' = \frac{1}{2}kr'^2 \Big|_0^r = \frac{1}{2}kr^2.$$

(taking $r=0$ to be the point of 0 potential energy)

kinetic energy:

$$T = \frac{1}{2}m|\dot{r}|^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2}, \quad (\text{where}$$

$$J = mX_0U_0)$$

$$\therefore E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r)$$

$$\therefore \text{effective potential energy} = \frac{J^2}{2mr^2} + V(r)$$

$$= \frac{m^2 X_0^2 U_0^2}{2mr^2} + V(r) = \boxed{\frac{mX_0^2 U_0^2}{2r^2} + \frac{1}{2}kr^2}$$

From initial conditions we know $E = \frac{1}{2}kX_0^2 + \frac{1}{2}mU_0^2$

$\therefore E = V(r) + \frac{1}{2}m\dot{r}^2$ - setting $\dot{r}=0$ to

find the limiting cases.

We have $E = V(r)$, which becomes

$$\frac{1}{2}kX_0^2 + \frac{1}{2}mU_0^2 = \frac{mX_0^2 U_0^2}{2r^2} + \frac{1}{2}kr^2.$$

$$\therefore (kx_0^2 + m\omega^2)r^2 = mx_0^2\omega^2 + kr^4$$

$$\therefore kr^4 - (kx_0^2 + m\omega^2)r^2 + mx_0^2\omega^2 = 0$$

$$\begin{aligned}\therefore r^2 &= \frac{(kx_0^2 + m\omega^2) \pm \sqrt{(kx_0^2 + m\omega^2)^2 - 4kmx_0^2\omega^2}}{2k} \\ &= \frac{kx_0^2 + m\omega^2 \pm \sqrt{(kx_0^2 - m\omega^2)^2}}{2k}\end{aligned}$$

$$\therefore r^2 = \frac{2kx_0^2}{2} = x_0^2$$

$$r_{3,4}^2 = \frac{2m\omega^2}{2k} = \frac{m\omega^2}{k} = \frac{\omega_0^2}{\omega^2}$$

$$\therefore r_1 = x_0, r_2 = -x_0, r_3 = \frac{\omega_0}{\omega}, r_4 = -\frac{\omega_0}{\omega}$$

in Solving the inequality $V(r) \leq E$ would give $\frac{\omega_0^2}{\omega^2} \leq r^2 \leq x_0^2$ for $x_0 \geq \frac{\omega_0}{\omega}$

or $x_0 \leq r^2 \leq \frac{\omega_0^2}{\omega^2}$ for $x \leq \frac{\omega_0}{\omega}$

The radial displacement r is limited by these inequalities. This shows that the system is bounded.

13.

$$\text{In (3)} \quad E = mr\dot{r}^2 + \frac{ml^2\dot{\theta}^2}{2r^2} - mgr - r$$

$$r_i = \frac{l}{2} \quad \dot{\theta}_i = \omega_0$$

$$\therefore E = mr\dot{r}^2 + \frac{ml^2\omega_0^2}{8r^2} + mgr - mgr$$

$$\begin{aligned}\therefore \frac{E}{mgr} &= \frac{\dot{r}^2}{g} + \frac{(l\omega_0)^2}{g l^2 r^2} + \frac{r}{l} - 1 \\ &= \frac{\dot{r}^2}{g} + \frac{\omega_0^2}{g} \frac{1}{r^2} + \frac{r}{l} - 1\end{aligned}$$

taking $\frac{E}{mgl} = \epsilon$, $\frac{r}{l} = \alpha$, $\frac{v_i^2}{gl} = \alpha$,

we have $\epsilon = \frac{r^2}{gl} + \frac{\alpha}{8x^2} + x - 1$.

$\therefore \epsilon = \frac{r^2}{gl} + U_e(x)$ (where $U_e = \frac{\alpha}{8x^2} + x - 1$)

(a)

Setting $0 = \frac{dU_e(x)}{dx} = \frac{-2\alpha}{8x^3} + 1$

$\therefore \frac{x^3}{4} = \alpha$ for x to be real

Also $E = \frac{1}{2}mv_i^2 - mgl(1 - l_i)$

$\therefore E = \frac{1}{2}mv_i^2 - mgl(1 - \frac{1}{2}) = \frac{1}{2}m(v_i^2 - gl)$

$\therefore \epsilon = \frac{(1/2)m}{mgl} (v_i^2 - gl) = \frac{1}{2}(\frac{v_i^2}{gl}) - \frac{1}{2} = \frac{\alpha}{2} - \frac{1}{2}$

To plot $U_e(x)$, we find the minimum value of

$U_e(x) = 0 \Rightarrow \frac{\alpha}{8x^2} + x - 1 = 0$

$\therefore \alpha + 8x^3 - 8x = 0$

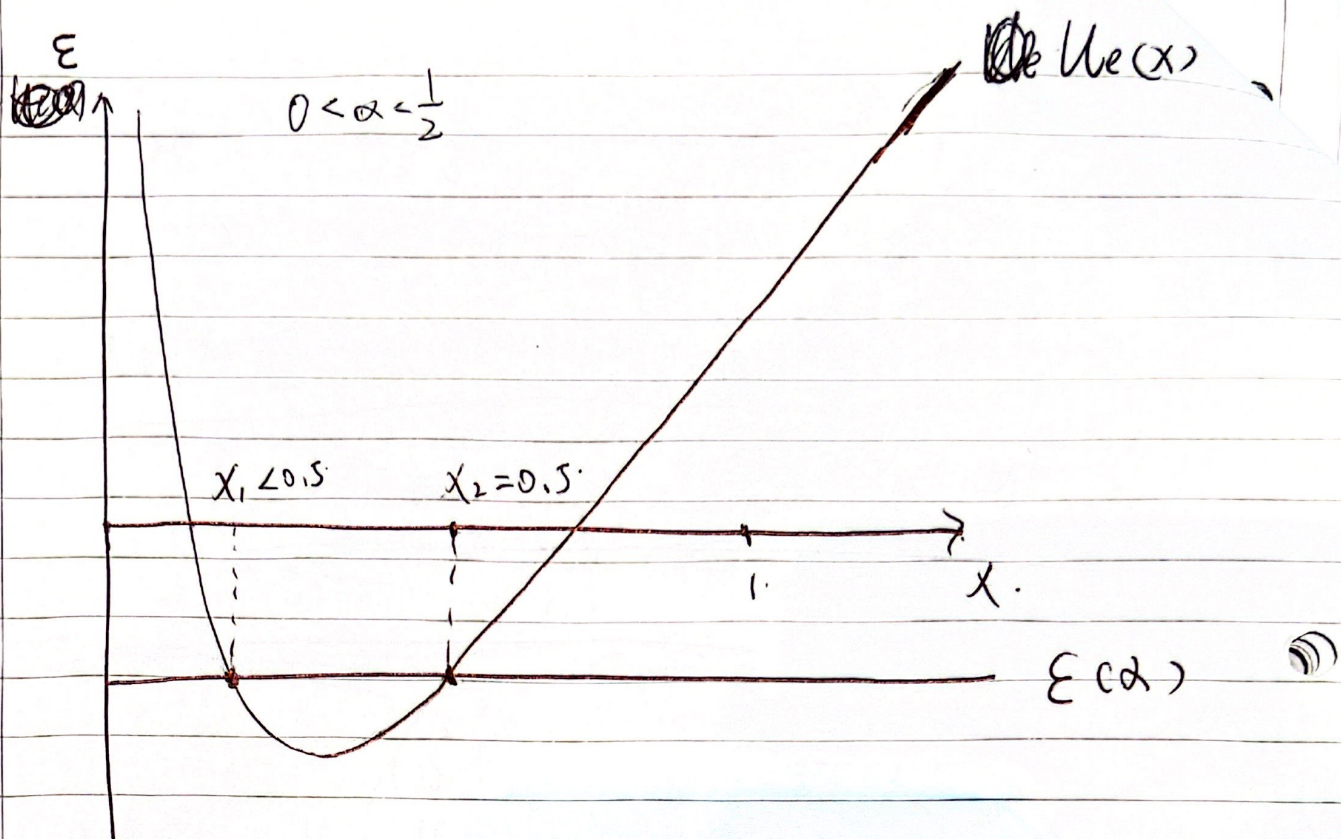
$U_e(x)$

Setting $\frac{dU_e(x)}{dx} = 0 \Rightarrow \frac{-2\alpha}{8x^3} + 1 = 0$

$\therefore x_{min} = \left(\frac{1}{4}\alpha\right)^{1/3}$

$U_e(x_{min}) = \frac{\alpha}{8(\frac{1}{4}\alpha)^{2/3}} + \left(\frac{1}{4}\alpha\right)^{1/3} - 1 = \frac{\alpha^{1/3}}{8} + \frac{\alpha^{1/3}}{4} - 1$

$= \frac{\alpha^{1/3}}{8} + \frac{\alpha^{1/3}}{4} - 1$
 $= \left(\frac{\sqrt[3]{16}}{8} + \frac{1}{4}\right)\alpha^{1/3} - 1$



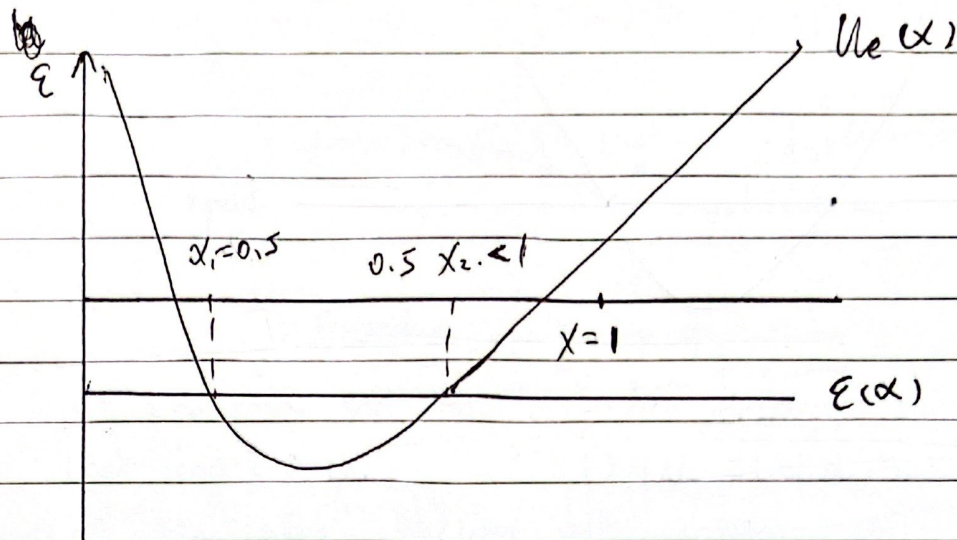
~~The motion of the particle is confined~~

$$\therefore E(\alpha) \geq U_e.$$

$\therefore$ The motion of the particle is confined within the region ~~$x_1 \leq x \leq x_2$~~ $x_1 \leq x \leq x_2 = 0.5$

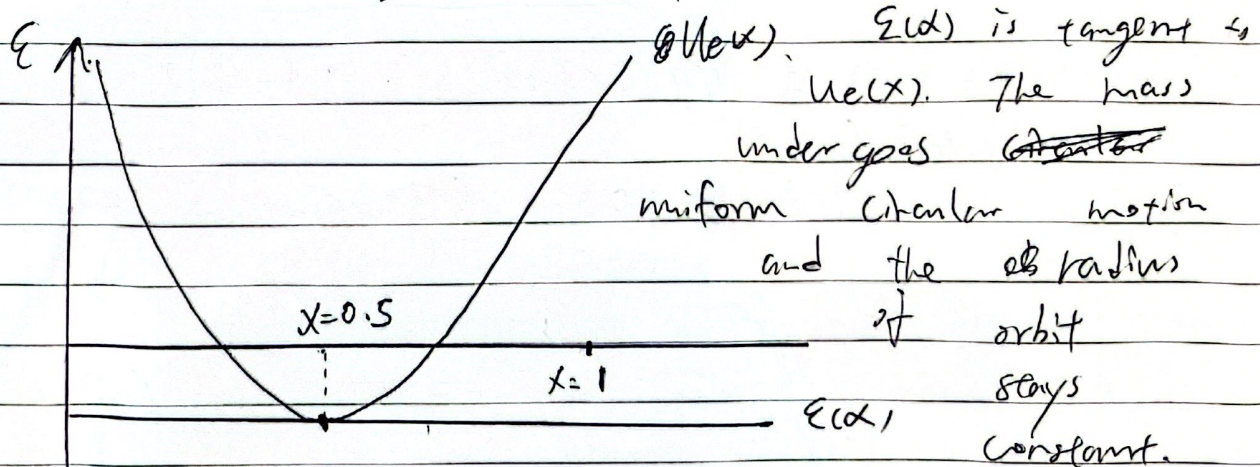
The mass starts moving ^{at $x=0.5$} and then the radius of the orbit decreases, then reaches a minimum and then increases, ^{then} reaches ~~$x=0.5$~~ and comes back to maximum $x_2=0.5$ the decreases again. The mass never goes above ~~$x=0.5$~~ and oscillates between ~~x_1~~ x_1 and x_2 .

(b)

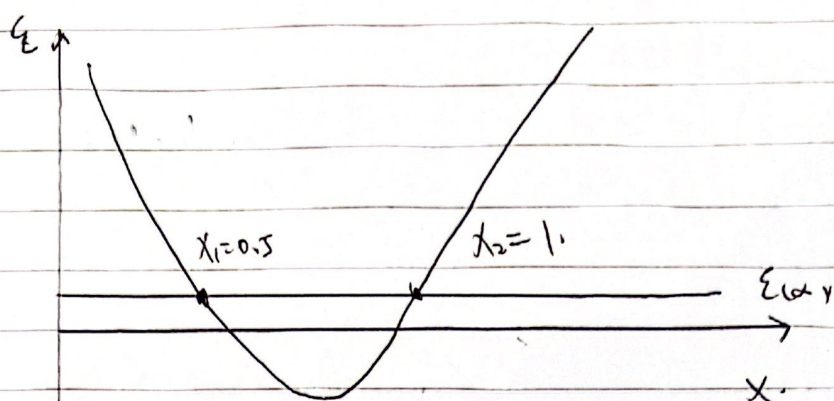
When $\frac{1}{2} < \alpha < \frac{4}{3}$ 

In this case the ~~particle~~ mass first goes outward, The radius of its orbit first increases and then decreases. The maximum radius of orbit is x_2 ($0.5 < x_2 < 1$) and its minimum is $x_1 = 0.5$. The mass never goes ~~below~~ below ~~$x=0.5$~~ $x=0.5$ and oscillates between x_1 and x_2 .

(c) When $\alpha = \frac{1}{2}$, the only possible value is $x=0.5$,



$E(x)$ is tangent to $U_e(x)$. The mass undergoes ~~circular~~ uniform circular motion and the radius of orbit stays constant.



When $\alpha = \frac{4}{3}$, the two solutions to $E(\alpha) = U(x)$ is $x_1 = 0.5$ and $x_2 = 1$.

This means the masses have just enough energy to for the bottom mass to reach and pass through the hole.

When $\alpha > \frac{4}{3}$, the real bottom mass will pass through the hole and the whole system will fly out of the table.

General expression for acceleration in polar coordinates

$$\vec{r}'' = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

Central force:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \Rightarrow r^2\dot{\theta} = \text{constant} \Rightarrow \dot{\theta} \text{ conserved.}$$

For bottom mass:

$$mg - T = -m\ddot{r} \quad (1)$$

For top mass

$$T = -m\ddot{r} + r\dot{\theta}^2 m \quad (2)$$

$$(1) + (2) \Rightarrow T = \frac{mg + r\dot{\theta}^2 m}{2}$$

$$\therefore T = \cancel{mg + \frac{J^2}{2r^3}}$$

$$\therefore J = mr^2\dot{\theta} \quad \therefore J^2 = m^2r^4\dot{\theta}^2$$

$$T = \frac{mgt + mr\dot{\theta}^2}{2} = \frac{mg + \frac{m^3r^4\dot{\theta}^2}{m^2r^3}}{2} = \cancel{mg}$$

$$= \frac{mg + \frac{mJ^2}{m^2r^3}}{2} = \frac{mg}{2} + \frac{J^2}{2mr^3}$$

$$J = m\frac{d}{dt}v\omega \quad \therefore J^2 = \frac{m^2d^2v^2}{4}$$

$$\therefore T = \frac{mg}{2} + \frac{\frac{m^2d^2v^2}{4}}{8mr^3g} \text{ etc.}$$

$$= \frac{mg}{2} + \frac{d^3}{8r^3}mg = \boxed{\left(\frac{1}{2} + \frac{d}{8x^3}\right)mg}$$

When $d = \frac{1}{2}$, $x = \frac{1}{2}$ tension $T = mg$, which corresponds to circular motion.

$$U(x) = \frac{d}{8x^2} + x - 1 \quad \text{and} \quad U(x) \leq U(x_1) = \frac{d}{2} - \frac{1}{2}$$

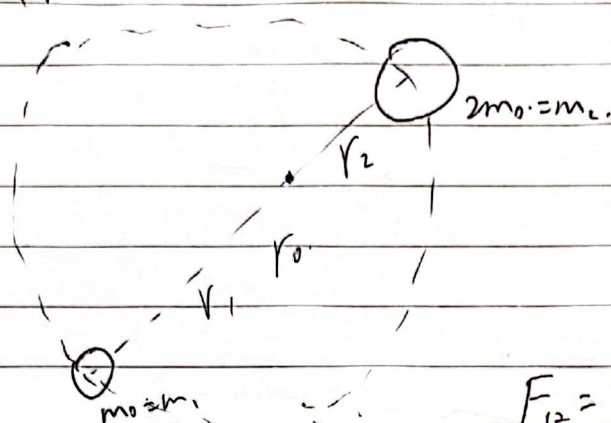
$$\therefore \lim_{x \rightarrow 0^+} U(x) = \infty \quad \text{for } d > 0$$

$\therefore$ the value of d cannot ~~approach~~ approach 0 from the positive side. when $d > 0$

$\therefore$ Only possible value for mass on the table to pass through the hole is.

$$\boxed{d=0}$$

14.



$$(a) \quad r_1 + r_2 = r_0$$

$$m_1 r_1 = m_2 r_2$$

$$\therefore r_1 = \frac{m_2}{m_1 + m_2} r_0 = \frac{2}{3} r_0$$

$$r_2 = \frac{m_1}{m_1 + m_2} r_0 = \frac{1}{3} r_0$$

$$F_{12} = \frac{G m_1 m_2}{r_0^2} = \frac{v_1^2}{r_1} m_1 = \frac{v_2^2}{r_2} m_2 \quad (1)$$

$$(1) \Rightarrow \therefore \frac{1}{2} m_1 v_1^2 = \frac{r_1 G m_1 m_2}{r_0^2} \quad \frac{1}{2} m_2 v_2^2 = \frac{r_2 G m_1 m_2}{r_0^2}$$

$$\therefore T = \frac{1}{2} m (v_1^2 + v_2^2) = \frac{(r_1 + r_2) G m_1 m_2}{r_0^2} = \frac{r_0 G m_1 m_2}{r_0^2} = \frac{G m_1 m_2}{2 r_0}$$

$$V_2 = -\frac{G m_1 m_2}{r_0} \quad \therefore E = T + V = \frac{-G m_1 m_2}{2 r_0} = -\frac{2 G M_0^2}{2 r_0} = \boxed{\frac{-G m_0^3}{r_0}}$$

$$(1) \Rightarrow m_1^2 v_1^2 = \frac{r_1 G m_1^2 m_2}{r_0^2} \quad m_2^2 v_2^2 = \frac{r_2 G m_1 m_2^2}{r_0^2}$$

$$\therefore J_1^2 = m_1^2 v_1^2 r_1^2 = \frac{r_1^3 G m_1^2 m_2}{r_0^2} \Rightarrow J_1 = \frac{r_1}{r_0} \sqrt{G m_1^3 m_2 r_1}$$

$$J_2^2 = m_2^2 v_2^2 r_2^2 = \frac{r_2^3 G m_1 m_2^2}{r_0^2} \Rightarrow J_2 = \frac{r_2}{r_0} \sqrt{G m_1 m_2^3 r_2}$$

$$J = J_1 + J_2 \quad \because m_1 r_1 = m_2 r_2 \quad \therefore G m_1^3 m_2 r_1 = G m_1 m_2^3 r_2$$

$$\therefore J = \frac{r_1 + r_2}{r_0} \sqrt{G m_1^3 m_2 r_1} = \sqrt{G m_1^3 m_2 r_1}$$

$$= \boxed{\sqrt{\frac{G m_1^3 m_2^3 r_0}{(m_1 + m_2)}}} = \sqrt{\frac{G (4 m_0^3) r_0}{3 m_0}} = \boxed{\frac{2}{3} \sqrt{G m_0^3 r_0}}$$

$$F_{12} = \frac{4\pi^2}{T^2} m_1 r_1 = \frac{G m_1 m_2}{r_0^2} = \frac{4\pi^2}{T^2} m_2 r_2$$

$$\therefore T = \sqrt{\frac{4\pi^2}{G m_2} r_0^3}$$

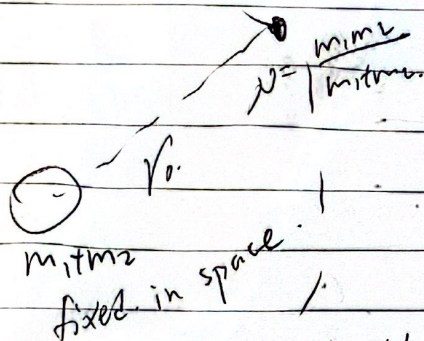
$$\therefore T = \frac{2\pi}{\sqrt{G m_2}} \sqrt{r_1 r_0^2} = \frac{2\pi r_0}{\sqrt{G m_2}} \sqrt{\frac{m_2}{m_1 + m_2}}$$

$$= \frac{2\pi}{\sqrt{G(m_1 + m_2)}} r_0^{3/2}$$

$$= \frac{2\pi}{\sqrt{3 G m_0}} r_0^{3/2}$$

(b)

Equivalent single particle ~~orbit~~ system that describe the relative motion of two ~~planets~~ stars.



relative velocity

$$v_0 = \sqrt{\frac{G m_1 m_2}{r_0}} + \sqrt{\frac{G m_2}{m_1 + m_2} r_0} = \sqrt{\frac{G(m_1 + m_2)}{r_0}}$$

$\therefore$ No CM motion

$\therefore$ Total Energy / Angular momentum is the Energy / Angular momentum relative to centre of mass

$$\therefore E = -\frac{1}{2} \mu v_0^2 = -\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \left(\frac{G(m_1 + m_2)}{r_0} \right)^2$$

$$= -\frac{G m_1 m_2}{2 r_0}$$

$$= -\frac{G m_0^2}{r_0}$$

$$J = \frac{G(m_1+m_2)r_0}{v_0} = \frac{Gm_1m_2}{\int \frac{G(m_1+m_2)}{r_0}} = \frac{Gm_1^2m_2^2r_0}{(m_1+m_2)} = \left[\frac{2}{3} \sqrt{3} \sqrt{Gm_0^3 r_0} \right]$$

$$\frac{G(m_1+m_2)r_0}{r_0^2} = \frac{4\pi^3}{T^2} r_0 \cdot v_0$$

$$\therefore \frac{T^2}{r_0^3} = \frac{4\pi^2}{G(m_1+m_2)}$$

$$T^2 = \frac{2\pi}{\sqrt{G(m_1+m_2)}} r_0^{3/2} = \frac{2\pi}{\sqrt{3Gm_0}} r_0^{3/2}$$

(c)

Assuming the explosion makes a loss of Δm of m_2 .

$\therefore$ The explosion is spherically symmetrical
 $\therefore$ velocity v_2 stays the same after explosion
 and hence relative velocity v_0 the instant.

$$m_2' = m_2 - \Delta m$$

$$N' = \frac{(m_2 - \Delta m)m_1}{m_2 + m_1 - \Delta m}$$

For the system to remain intact.

$$\frac{1}{2} N' v_0^2 + V' < 0$$

where $V = \frac{G m_1 m_2'}{r_0} = - \frac{G m_1 (m_2 - \Delta m)}{r_0}$

$$\therefore \frac{1}{2} \frac{m_1 (m_2 - \Delta m)}{m_1 + m_2 - \Delta m} \cdot \frac{G (m_1 + m_2)}{r_0} < \frac{G m_1 (m_2 - \Delta m)}{r_0}$$

$$\therefore \frac{m_1 + m_2}{2} > m_1 + m_2 - \Delta m$$

$$2(m_1 + m_2 - \Delta m) > m_1 + m_2$$

$$\therefore \Delta m < \frac{1}{2} (m_1 + m_2) \text{ is the}$$

condition for the system to remain intact.

In this case. $m_1 = \Delta m = m_0$. $m_2 = 2m_0$.

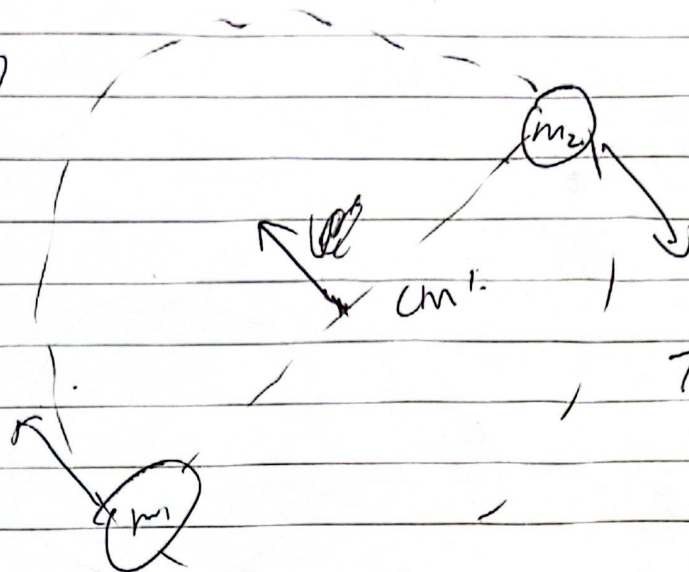
$$\therefore \frac{1}{2} (m_1 + m_2) = \frac{3}{2} m_0 > m_0 = \Delta m.$$

$\therefore$ The condition is satisfied.

$\therefore$ The system will remain intact.

Actual binary orbit.

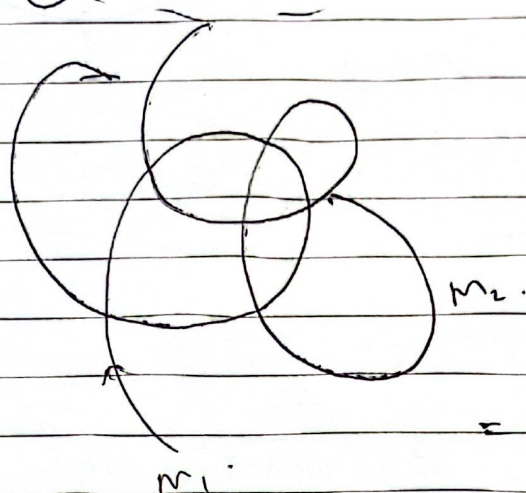
(d)



The velocity of centre of mass is no longer zero after collision

It is equal to

$$V_c = \frac{m_1 V_1 - (m_2 - \Delta m) V_2}{m_1 + m_2 - \Delta m}$$



$$= \frac{(m_1 V_1 - m_2 V_2) + \Delta m V_c}{m_1 + m_2 - \Delta m}$$

$$\therefore m_1 V_1 - m_2 V_2 = 0$$

$$\therefore V_c =$$

$$\boxed{\frac{\Delta m V_2}{m_1 + m_2 - \Delta m}}$$

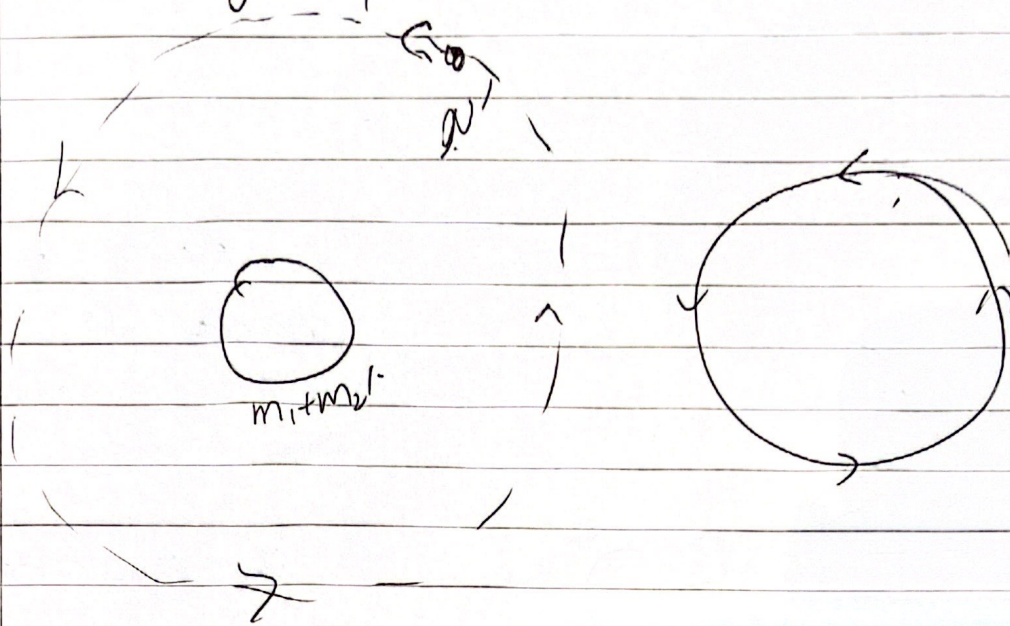
$$= \frac{m_0}{m_1 + m_2 - m_0} V_2 = \frac{1}{2} V_2$$

$$V_2 = \sqrt{\frac{G m_1 r_0}{r_0^2}} = \sqrt{\frac{G m_1}{r_0^2} \frac{m_1}{m_1 + m_2}}$$

$$= m_1 \sqrt{\frac{G}{r_0 (m_1 + m_2)}} = m_0 \sqrt{\frac{G}{3 m_0 r_0}} = \sqrt{\frac{G m_0}{3 r_0}}$$

$$\therefore V_c = \boxed{\frac{1}{2} \sqrt{\frac{G m_0}{3 r_0}}}$$

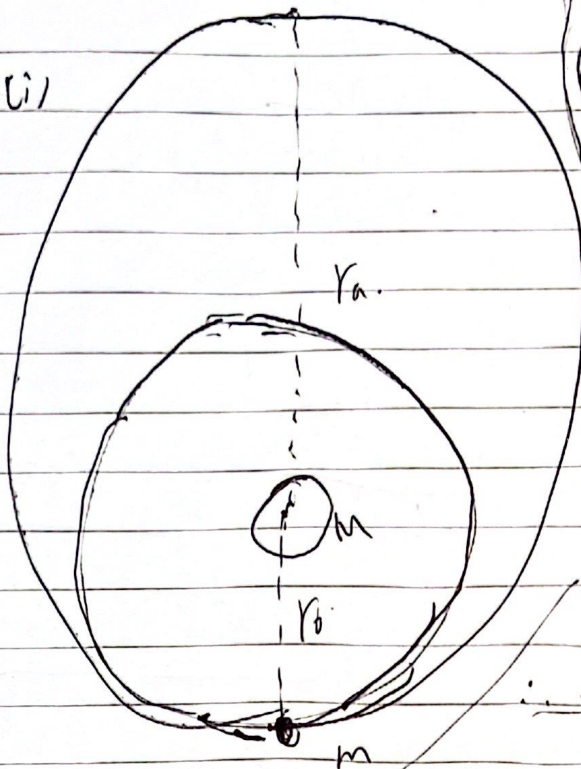
Single particle orbit.



Conservation of Energy

15. (i)

Conservation of energy.



$$\frac{1}{2} m v_0^2 - \frac{G M m}{r_0} = \frac{1}{2} m v_p^2 - \frac{G M m}{r_p}$$

$$v_0 = \sqrt{\frac{G M}{r_0}}$$

$a = \frac{r_a + r_0}{2}$ for elliptical orbit.

$$\frac{1}{2} m v_p^2 - \frac{G M m}{r_0} = \frac{G M m}{r_a + r_0}$$

$$v_p^2 = \frac{G M}{r_0} - \frac{2 G M}{r_a + r_0}$$

$$= G M \left(\frac{1}{r_0} - \frac{2}{r_a + r_0} \right)$$

$$= G M \left(\frac{r_a + r_0 - 2 r_0}{r_0 (r_a + r_0)} \right) = G M \left(\frac{r_a - r_0}{r_0 (r_a + r_0)} \right)$$

$$\Rightarrow v_p^2 = G M \left(\frac{2}{r_0} - \frac{2}{r_a + r_0} \right)$$

$$= G M \left(\frac{2 r_a + 2 r_0 - 2 r_0}{r_0 (r_a + r_0)} \right) = G M \left(\frac{2 r_a}{r_0 (r_a + r_0)} \right)$$

$$\therefore \left(\frac{v_p}{v_0} \right)^2 = \frac{G M \left(\frac{2 r_a}{r_0 (r_a + r_0)} \right)}{G M \left(\frac{1}{r_0} \right)} = \frac{2 r_a}{r_a + r_0}$$

(ii) $r_a = r_m = 60 r_e$

$$\frac{2 r_a}{r_0 + r_a} = \frac{2 \times 60}{60 + 1} = \frac{120}{61} \approx 1.967$$

$$v_0 = \sqrt{\frac{G M_e}{r_0^2}} = \sqrt{\frac{6.62 \times 10^{-11} (5.57 \times 10^{24})}{(6.38 \times 10^6)^2}} = 7900 \text{ m/s}$$

$$\therefore v_p = 1.403 \times v_0 = 1.403 \times 7900 \text{ m/s} = 1.108 \times 10^4 \text{ m/s}$$

$$r_a = r_m$$

$$(ii) \text{ If } \frac{V_p}{V_0} = \left(\frac{V_p}{V_0} \right)^2 = \frac{2r_a}{r_a + r_0} = \frac{2 \times 60}{61}$$

$$\therefore \frac{V_p}{V_0} = \sqrt{\left(\frac{2 \times 60}{61} \right)^2} = \underline{1.403}$$

$$V_0 = \sqrt{\frac{GM_e}{R_e}} = \sqrt{\frac{(6.67 \times 10^{-11}) (5.97 \times 10^{24})}{(6.38 \times 10^6)}} = \underline{7900 \text{ m/s}}$$

$$V_p = 1.403 \times 7900 = \underline{1.108 \times 10^4 \text{ m/s}}$$

(iii)

$$\text{New } \frac{V_p}{V_0} = 0.99 \sqrt{\frac{2 \times 60}{61}} = 1.389$$

$$\therefore \left(\frac{V_p}{V_0} \right)^2 = 1.928 = \frac{2r_a}{r_0 + r_a}$$

$$\therefore (r_0 + r_a) 1.928 = 2r_a$$

$$\therefore r_a (2 - 1.928) = 1.928 r_0$$

$$\therefore \cancel{r_a} \approx 27 r_0$$

$$r_a \approx 27 r_0 = 27 R_e \approx 30 R_e \approx \frac{1}{2} r_m$$

$\therefore$ Approximately half of the radius of moon's orbit has been missed!