EM 4 Ziyan Li

F.0.

For a surface S that spans a closed curve C, The Bagnetic flux density $\vec{B}$, Induced emf $\mathcal{E}$, induced electric field $\vec{E}$ and magnetic flux through S is related by

$$\mathcal{E} = \int_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = -\frac{d\Phi}{dt}.$$

This is Faraday's Law

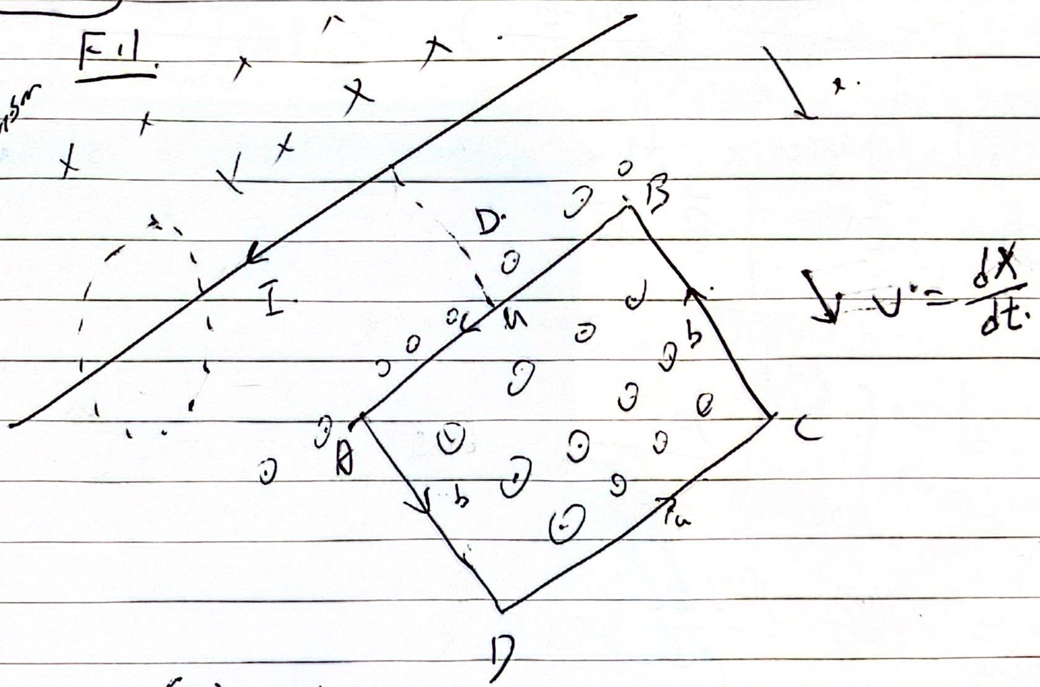
Self inductance is the property of conductor by which change in its current induces a voltage in itself

Mutual inductance is that by which change in current induces a voltage in ~~the~~ another conductor.

Lenz's law states that the direction of the induced electromotive force is such that the induced current creates a magnetic field that opposes the change in magnetic flux.

F.1.

W. J. Duffin: Electromagnetism



(a) (i)

AC and BD do not cut magnetic field lines so induce no emf.
Emf induced ~~on~~ on AB and CD oppose each other.

$\cancel{\varepsilon_{AB} = Blv = Ba}$

Magnetic flux density B is given by Ampere's law. For a point a distance x away ~~from the~~ from the long wire

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I. \qquad \therefore (2\pi x)B = \mu_0 I.$$

$$\therefore \cancel{B = \frac{\mu_0 I}{2\pi x}} \qquad B(x) = \frac{\mu_0 I}{2\pi x}$$

$$\varepsilon_{AB} = Blv = Bav = B(D)\,av = \frac{\mu_0 I a v}{2\pi D},$$

$$\varepsilon_{CD} = Blv = B(D+b)\,av = \frac{\mu_0 I a v}{2\pi (D+b)}$$

$$\varepsilon = \varepsilon_{AB} - \varepsilon_{CD} = \frac{\mu_0 I a v}{2\pi}\left(\frac{1}{D} - \frac{1}{(D+b)}\right)$$

$$= \frac{\mu_0 I a v}{2\pi}\left(\frac{D+b-D}{D(D+b)}\right) = \boxed{\frac{\mu_0 I v a b}{2\pi D(D+b)}}$$

(ii) Magnetic flux when AB is a ~~distance~~ distance x from wire

~~Magnetic flux~~ at

$$\Phi(x) = \int \vec{B}\cdot d\vec{A} = \int_x^{x+b} B(x)\cdot d\vec{A}$$

$$= \int_x^{x+b} \left(\frac{\mu_0 I}{2\pi x}\right)(a\,dx) = \frac{\mu_0 I a}{2\pi}\int_x^{x+b}\frac{dx}{x}.$$

$$= \frac{\mu_0 I a}{2\pi}\ln\left(1+\frac{b}{x}\right)$$

value ~~Magnitude~~ of emf:

$$\varepsilon(x) = -\frac{d\Phi(x)}{dt} = -\frac{d\Phi}{dx}\frac{dx}{dt} = v\frac{d\Phi}{dx} = \frac{\mu_0 I a v}{2\pi}\frac{d}{dx}\left(\ln\left(1+\frac{b}{x}\right)\right)$$

$$= -\frac{\mu_0 I a v}{2\pi}\frac{1}{1+\frac{b}{x}}\left(-\frac{b}{x^2}\right) = \frac{\mu_0 I a v}{2\pi}\frac{b}{x^2+bx}$$

Magnitude of emf at $x = D$

$$\cancel{\mathcal{E} = \mathcal{E}}$$

$$\mathcal{E} = |\mathcal{E}(D)| = \frac{\mu_0 I a v}{2\pi} \frac{b}{D^2 + bD}$$

$$= \boxed{\frac{\mu_0 I v a b}{2\pi D (D+b)}}$$

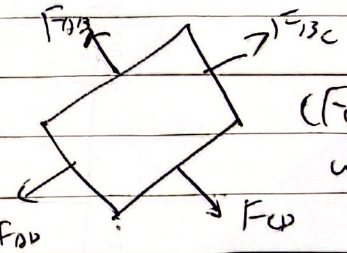
(b) If resistance is low than the induced current will be high and thus the magnetic field produce by this current itself cannot be neglected. anymore. ($\mathcal{E} = IR$)

We know $\mathcal{E}$ is depend on the distance the loop is away from the wire so $\mathcal{E}$ is changing $\because \mathcal{E} = IR$ $\therefore I$ is changing and the magnetic flux. it produces is changing. ~~So the~~ This change in magnetic field will affect the net value of $\frac{d\Phi}{dt}$ and thus our result might be different.

(c) When coil is moving with constant speed v, the ~~force~~ external force and lorentz force are ~~both~~ balanced.

$\therefore F = F_B$.

Current on the coil



(F_{BC} ~~cancels~~ cancels with F_{AD})

$$I' = \frac{\mathcal{E}}{R} = \frac{\mu_0 I v a b}{2\pi R D (D+b)}$$

$F_B = \ell I B \ell = F_{AB} - F_{CD}$

$$= I' a \left(B(D) - B(D+b) \right) = \frac{\mu_0 I v a b}{2\pi R D (D+b)} \cdot a \left(\frac{\mu_0 I}{2\pi} \right) \left(\frac{1}{D} - \frac{1}{D+b} \right)$$

$$= \frac{\mu_0^2 I^2 v a^2 b}{4\pi^2 R D(D+b)} \left[\frac{D+b-D}{D(D+b)} \right]$$

$$= \boxed{\frac{\mu_0^2 I^2 v a^2 b^2}{4\pi^2 R D^2 (D+b)^2}}$$

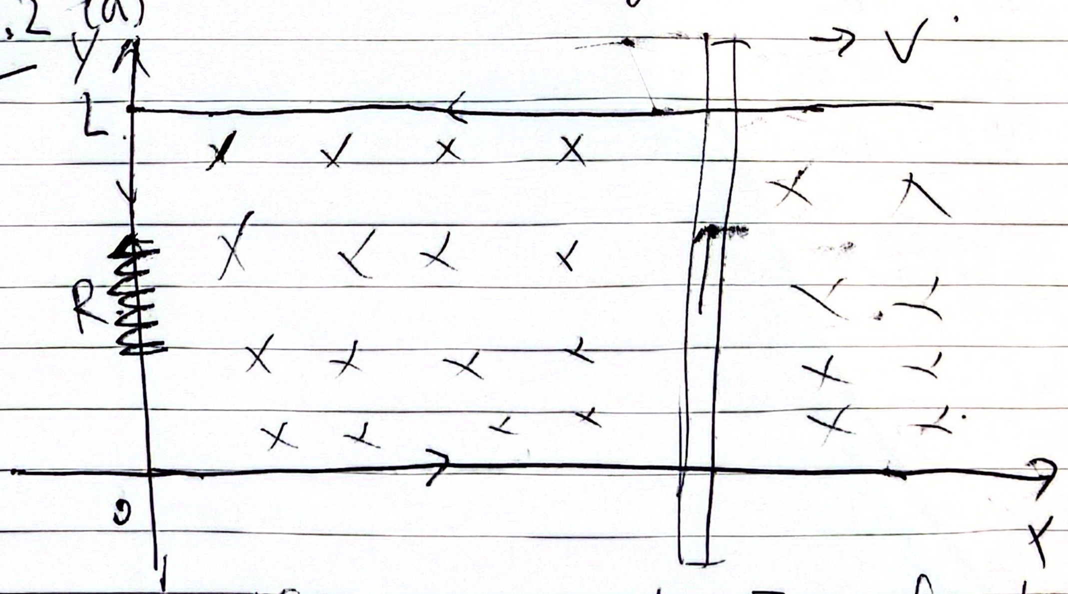
Mechanical Power $\quad P = Fv = \boxed{\frac{(\mu_0 I vab)^2}{4\pi^2 R D^2 (D+b)^2}}$

Rate of Generation of Heat : $\quad \frac{dE}{dt} = I'^2 R$

$$= \left(\frac{\mu_0 I vab}{2\pi R D(D+b)} \right)^2 \cdot R = \boxed{\frac{(\mu_0 I vab)^2}{4\pi^2 R D^2 (D+b)^2}}$$

$\therefore P = \frac{dE}{dt}$, i.e. mechanical power equals to the rate of generation of heat.

E.2 (a)



$\cancel{\Phi = \frac{d}{dt}\int B \cdot dA}$ $\cancel{\Phi = \frac{d}{d}}$ $\Phi = \int B \cdot dA$

$$\varepsilon = \frac{d\Phi}{dt} = \frac{d}{dt}\int B \cdot dA = B \frac{d}{dt}\int dA = B \frac{dA}{dt}.$$

$$= B\frac{d(xL)}{dt} = BL\frac{dx}{dt} = BLv$$

$$\therefore I = \frac{\mathcal{E}}{R} = \boxed{\frac{BLv}{R}}$$ is the induced current

(b) External force F balances Lorentz's force F_B by $F = F_B$

$$F_B = IBL = \frac{BLv}{R} \cdot B \cdot L = \frac{B^2L^2v}{R}$$

$$\therefore \boxed{F = \frac{B^2L^2v}{R}}$$

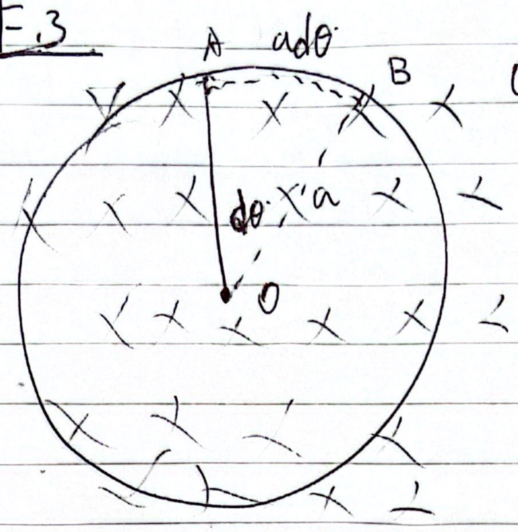
(c) power supplied $P = Fv = \boxed{\frac{B^2L^2v^2}{R}}$

(d) power dissipated in the resistor

$$P_2 = I^2R = \left(\frac{BLv}{R}\right)^2 R = \boxed{\frac{B^2L^2v^2}{R}}$$

We have $\underline{P_1 = P_2}$

E.3



(a)

Consider a closed loop C ~~drawn~~ (a sector) drawn on the left.
OB and $\widehat{AB}$ are imaginary lines
So $E_{AB} = E_{OB} = 0$ ∴ $\oint_C E\cdot dl = \mathcal{E}_{OA}$
Applying Faraday's law:

$$\oint_C E\cdot dl = -\frac{d}{dt}\int_{S_{OAB}} B\cdot dA = -\frac{d\Phi}{dt}$$ for constant B

$$\cancel{E_{OA} = \oint_C E\cdot dl = B\frac{d}{dt}(S_{OAB}) = B\frac{d}{dt}(}$$

$$d\Phi = B\cdot dA = B\left(\frac{1}{2}a\cdot a\,d\theta\right) = \frac{1}{2}Ba^2\,d\theta$$

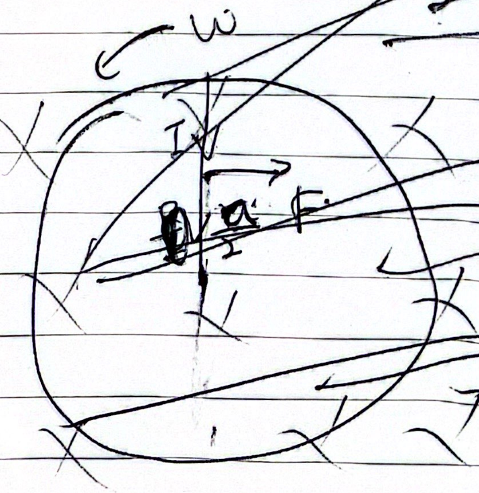
$$\therefore V = E_{OA} = \frac{d\Phi}{dt} = \frac{1}{2}a^2B\frac{d\theta}{dt} = \boxed{\frac{\omega a^2 B}{2}}$$

~~(b)~~

~~Initial value of current~~

$$\cancel{I = \frac{V}{R} = \frac{\omega^2 aB}{2R} = \frac{(3000/6.5)^2(3m)(0.5T)}{2(10^{-5}\Omega)}}$$

$$\cancel{= 1.875\times10^{6}A}$$



~~Force on disk $F = IBa$.~~
~~Torque on disk.~~
$\cancel{\tau = F\cdot\frac{a}{2} = \frac{IBa^2}{2}}$

~~Torque is opposing angular~~

(b) Initial value of current is i
then.

$i = \frac{V}{R}$ by ohm's ~~law~~ law

$\because V = \frac{\omega a^2 B}{2}$ $\therefore i = \frac{\omega a^2 B}{2R}$

$\because R = 10^{3}\Omega$. $\omega = 3000\,min^{-1} = 50\,s^{-1}$ $a = 3m$ $B = 0.5T$.

$$\therefore i = \frac{(50s^{-1})(3m)^2(0.5T)}{2(10^{-3}\Omega)} = \boxed{1.125\times10^{5}A}$$

Lorentz force on disk: ~~$F = IBa$~~ $=$

$$F = iBa = \left(\frac{\omega a^2 B}{2R}\right)(B)(a) = \frac{\omega a^3 B^2}{2R}$$

Torque on disk $\tau = -F\left(\frac{a}{2}\right) = \frac{\omega a^4 B^2}{4R}$

(negative sign indicates direction opposite to disk's rotation)

Moment of inertia of disk $I = \frac{1}{2}ma^2$

$\because \tau = I\ddot{\theta} = I\frac{d\omega}{dt}$ $\therefore \left(\frac{1}{2}ma^2\right)\frac{d\omega}{dt} = -\frac{a^4B^2}{4R}\omega$

$$\therefore \frac{d\omega}{dt} = -\frac{B^2a^2}{2mR}\omega$$

let t be the time for disk to slow to half its initial speed,

$\dot{\theta}$. initial speed $\omega_0 = 3000\,min^{-1} = 50s^{-1}$

$$\frac{d\omega}{\omega} = -\frac{B^2a^2}{2mR}\,dt.$$

$$\therefore \int_{\omega_0}^{\omega_0/2} \frac{d\omega}{\omega} = -\frac{B^2a^2}{2mR}\int_0^t dt'$$

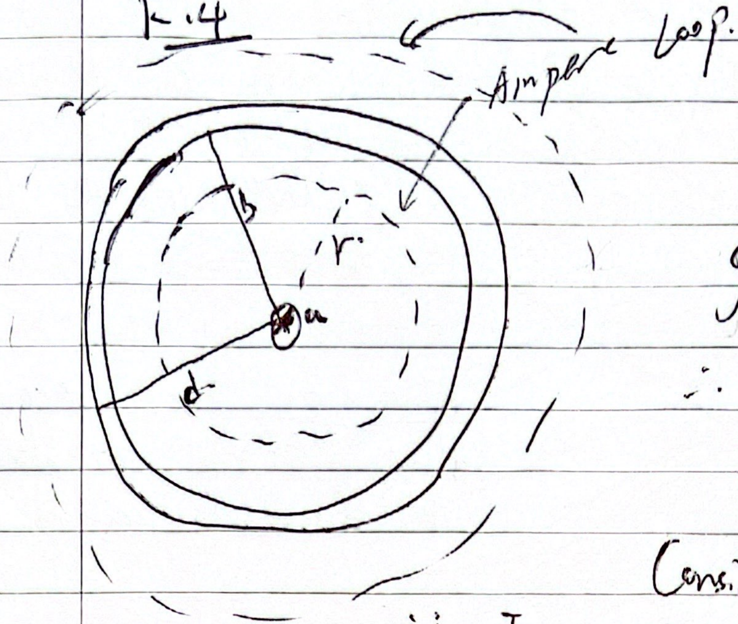
$$\therefore \ln\left(\frac{\omega_0}{2}\right) - \ln(\omega_0) = -\frac{B^2a^2}{2mR}\,t.$$

$$\therefore t\,\frac{B^2a^2}{2mR} = \ln\left(\frac{\omega_0}{\omega_0/2}\right) = \ln 2$$

$$\therefore t = 2\ln 2\,\frac{mR}{B^2a^2}$$

$$\therefore t = \frac{(2\ln 2)(10^4\,kg)(10^{-3}\,\Omega)}{(0.5\,T)^2(3\,m)^2} = \boxed{6.16\,s}$$

K.4



Consider Ampere loop $a<r<b$.

$$\oint B \cdot dl = \mu_0 I.$$

$$\therefore B(2\pi r) = \mu_0 I$$

$$\therefore B = \frac{\mu_0 I}{2\pi r}.$$

Consider Ampere loop $r>d$.

$\because$ In coax cable the current of inside and outside layers flow in opposite directions

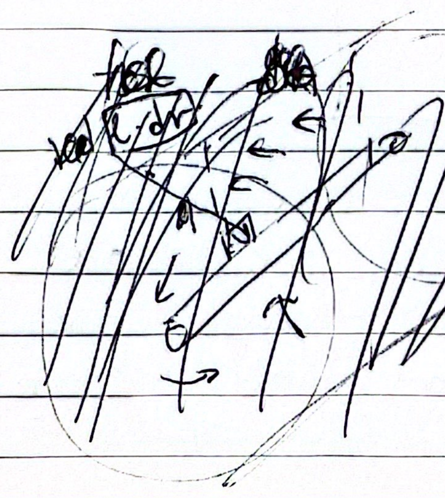
$\therefore$ penetrating current for this loop is 0

$$\therefore \oint B \cdot ds = \mu_0 I = 0 \quad \therefore B = 0 \cdot \text{ for } r>d$$

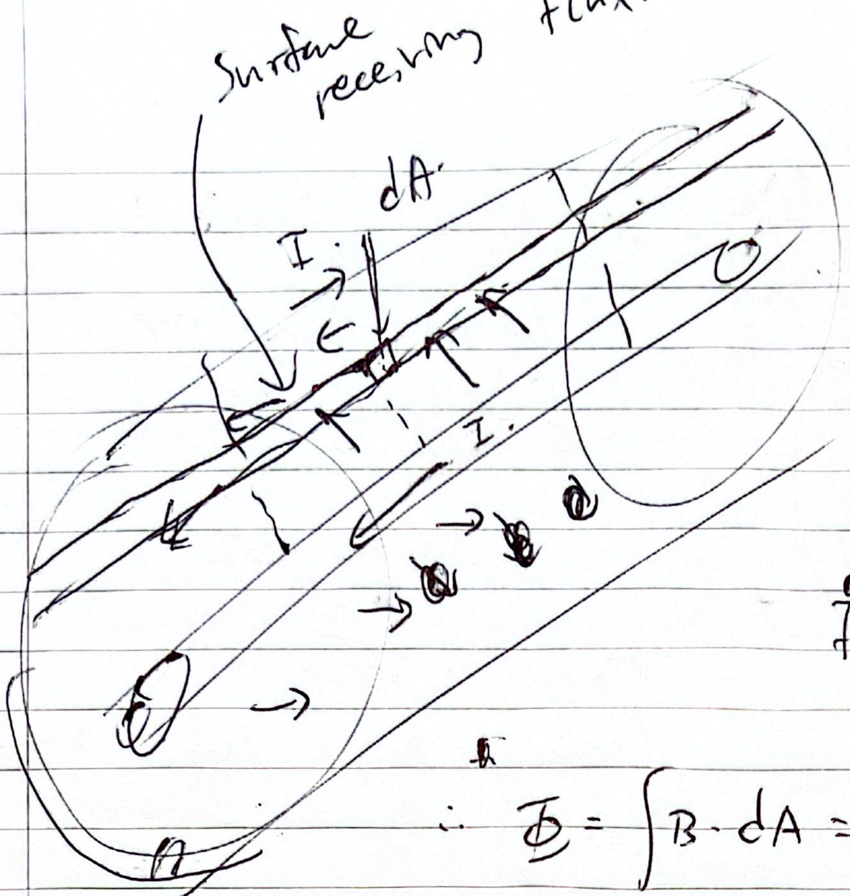
We ignore the flux through the wires since $a \ll b-a$ and $d-b \ll b-a$.

$$\therefore \Phi = \int B \cdot dA$$

we know $B = \frac{\mu_0 I}{2\pi r}$, for dA.



we consider the surface that receives magnetic flux, which is the surface that is parallel to and connecting the central cable and outer shield.



let l be the length of cable ($l \gg b-a$)

$\therefore$ we have long ~~dA~~ strips of differential surfaces $dA = l\,dr$ that receives flux $B(r)$ at different r from $r=a$ to $r=b$.

$$\therefore \Phi = \int B \cdot dA = \int_a^b \frac{\mu_0 I l}{2\pi r}\,dr$$

$$\Phi = \frac{\mu_0 I l}{2\pi} \ln \frac{b}{a}$$

$$\therefore L = \frac{\Phi}{I}$$

$$= \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)$$

$\therefore$ Self inductance per-length is

$$\boxed{\frac{L}{l} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)}$$

The assumptions $(b-a) \gg a$ and $(b-a) \gg (d-b)$ are necessary because in the above calculation we ignored the flux through the cables. Only we these conditions holds are the flux through the cables negligible

~~The assumptions~~ compare to ~~the~~ flux ~~B~~ between the cables.

If these conditions do not hold then the inner

~~Then inside the inner rod (r<a)~~

$$\cancel{\oint B_1 \cdot ds = \mu_0 \left(\frac{r^2}{a^2} I\right)}$$

$$\cancel{B(2\pi r) = \mu_0 I \frac{r^2}{a^2} \quad \therefore B_1 = \frac{\mu_0 I r}{2\pi a^2}}$$

$$\cancel{\Phi_1 = \int_0^a B\,dA = \int_0^a \frac{\mu_0 I r}{2\pi a^2} l\,dr = \frac{\mu_0 I l}{2\pi a^2}\int_0^a r^2\,dr}$$

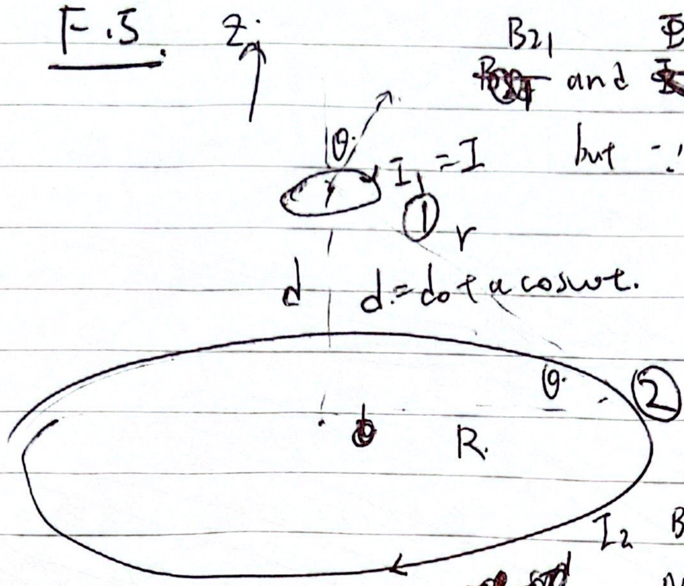
$$\cancel{= \frac{\mu_0 I a^2}{4\pi a^2} \quad \therefore L = \frac{\Phi_1}{I} = \frac{\mu_0 a^2 l}{4\pi a^2} = \frac{\mu_0 l}{4\pi}}$$

cable and outer ~~ski~~ conducting shield also contribute to the inductance. The ~~cotum~~ calculation of these inductances require the use of flux ~~leng~~ linkage

B_{21} is B-field produced by ~~Φ~~ 1 on 2

Φ_{21} is flux caused by 1 on 2.

F.5.



B_{21} and Φ_{21} are difficult to calculate

but $\because V = -\frac{d\Phi_{21}}{dt}$, and

$\Phi_{21} = I_1 M_{21}$ where

M is the ~~mutual~~ mutual inductance between 1 and 2.

By reciprocity ~~p~~ theorem

$M_{21} = M_{12}$.

$$M_{12} = \frac{\Phi_{12}}{I_2} = \cancel{\frac{N}{I_2}\int B_{12}\cdot dA} = \frac{1}{I_2}\int \vec{B_{12}}\cdot \vec{dA}$$

~~Since~~ One the axis of ring 2.

$d\vec{B}_{12} = \frac{\mu_0 I}{4\pi r^2} \vec{dl} \times \hat{r}$ ~~$B_{12} = \frac{\mu_0 I}{4\pi r^2}$~~ all other

directions of $\vec{B_{12}}$ cancels, only z-direction remains. So $\vec{B_{12}} = \frac{\mu_0 I_2}{4\pi r^2}(2\pi R)\cos\theta$.

$$= \frac{\mu_0 I_2}{4\pi(R^2+d^2)}(2\pi R)\left(\frac{R}{\sqrt{R^2+d^2}}\right) = \frac{\mu_0 I_2 R^2}{2(R^2+d^2)^{3/2}}.$$

$\therefore$ ~~$M_{12} = \frac{\Phi_{12}}{I_2}$~~

$\because$ coil 1 is small compare to coil 2

$\therefore$ The magnetic field through 1 is nearly uniform

$$\therefore \Phi_{12} = \vec{B_{12}}\cdot A \cdot N = \frac{\mu_0 N A I_2 R^2}{2(R^2+d^2)^{3/2}}$$

$$\therefore M_{12} = \frac{\Phi_{12}}{I_2} = \frac{\mu_0 N A R^2}{2(R^2+d^2)^{3/2}}.$$

$$V = -\frac{d\Phi_{21}}{dt} = -\frac{d(IM_{21})}{dt} = -\frac{d(IM_{12})}{dt}.$$

$$= -\left(I\frac{dM_{12}}{dt} + M_{12}\frac{dI}{dt}\right)$$

$\because$ constant current I $\therefore$ $\frac{dI}{dt} = 0.$

$$\therefore V = -I\frac{dM_{12}}{dt} = -I\frac{d}{dt}\left(\frac{\mu_0 N A R^2}{2(R^2+d^2)^{3/2}}\right)$$

$$= -\frac{\mu_0 N A I R^2}{2}\left[\frac{d}{dt}\left((R^2+d^2)^{-3/2}\right)\right].$$

$$= -\frac{\mu_0 N A I R^2}{2}\left(-\frac{3}{2}(R^2+d^2)^{-5/2}\right)(2d)\left(\frac{d(d)}{dt}\right)$$

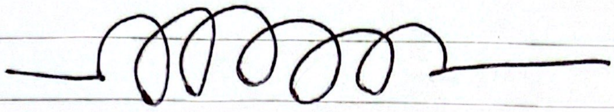
$$= +\frac{3}{2}\mu_0 N A I R^2 d\,(R^2+d^2)^{-5/2}\left[\frac{d}{dt}\left(d_0 + a\cos(\omega t)\right)\right]$$

$$= +\frac{3}{2}\mu_0 N A I\frac{R^2 d}{(R^2+d^2)^{5/2}}\left(-\omega a\sin\omega t\right)$$

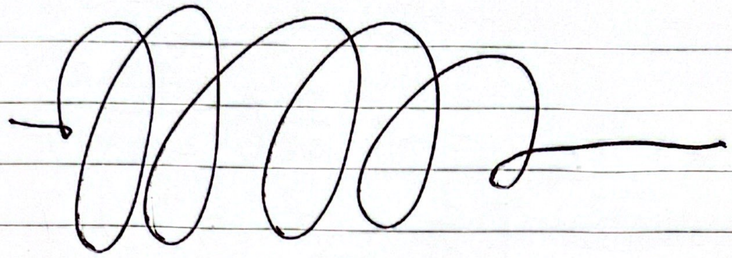
take the magnitude of V we have:

$$\boxed{V = \frac{3}{2}\mu_0 N A I\omega\frac{aR^2 d}{(R^2+d^2)^{5/2}}\sin\omega t.}$$

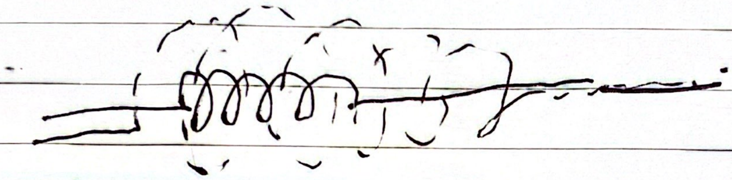
F.6



① r=a



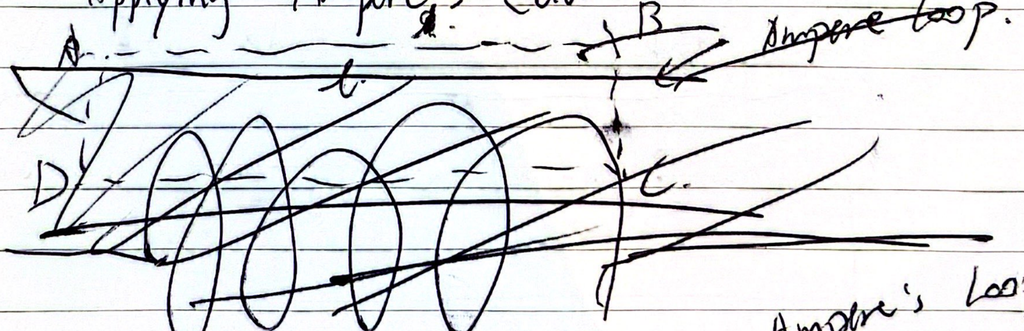
② r=2a.

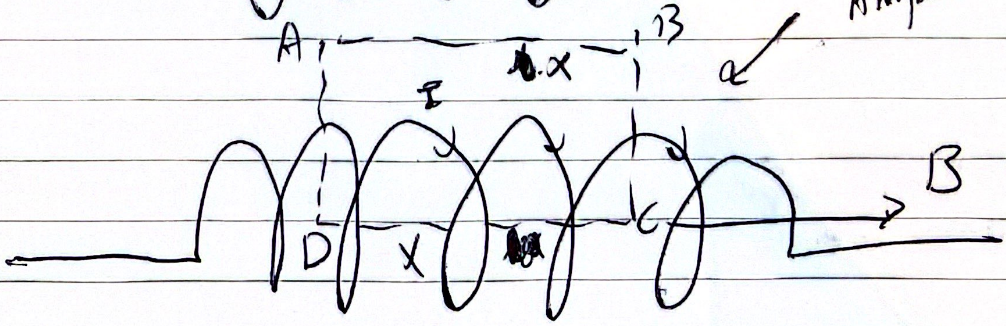


Mutual

(a) At any point far from the ends inside a solenoid the magnetic field is approximately constant

∴ Applying Ampere's Law:





Ampere's Loop ABCD

AB = CD = x

Consider the Ampere's loop ABCD:

Along CD ~~the~~ $\int_{CD} \vec{B}\cdot d\vec{s} = BX$

And we know that for a long solenoid the field inside is constant and field outside is 0.

$\therefore \int \vec{B}\cdot d\vec{s}$ for AB is $\int_{AB} \vec{B}\cdot ds = 0.$

For ~~AB~~ BC and AD, $\vec{B} \perp d\vec{s}$

$\therefore \int_{AD} \vec{B}\cdot d\vec{s} = \int_{BC} \vec{B}\cdot d\vec{s} = 0$

$\therefore \int \vec{B}\cdot d\vec{s} = \mu_0 I_p$ (I_p is current penetrating)

$\therefore BX = \mu_0 I_p$.

$\because$ The solenoid has n turns per unit length.

$\therefore$ there are nx turns inside the loop.

$\therefore I_p = nx I$

$\therefore Bx = \mu_0 n x I$ $\therefore$ ~~BE~~

$\therefore \boxed{B = \mu_0 n I}$

(b)

Self inductance of ①

$A_1 = N\pi a^2$ $\Phi_1 = B\pi a^2 = \mu_0 n I \pi a^2 N = \mu_0 I \pi a^2 \frac{N^2}{l}$

$\therefore L_1 = \frac{\Phi_1}{I} = \boxed{\mu_0 \pi a^2 \frac{N^2}{l}}$

Self inductance of ②.

$A_2 = N\pi(2a)^2 = 4N\pi a^2$. $\Phi_2 = BA_2 = 4N\pi a^2 \mu_0 I \frac{N}{l}$.

$$L_2 = \frac{\Phi_2}{I} = \cancel{4N\pi a^2}\ \boxed{4\pi a^2 \mu_0 \frac{N^2}{l}}$$

Mutual inductance:

Effect of field of ② on ①:

$M_{12} = \frac{\Phi_{12}}{I_2}$. $\Phi_{12} = BA = (\mu_0 I_2 \frac{N}{l})(\pi a^2 N)$

$$\therefore M_{12} = \frac{\Phi_{12}}{I_2} = \boxed{\pi a^2 \mu_0 \frac{N^2}{l}}$$

Effect of field of ① on ②.

$M_{21} = \frac{\Phi_{21}}{I_1}$ $\Phi_{21} = BA = (\mu_0 I_1 \frac{N}{l})(\pi a^2 N)$

effective area

$$M_{21} = \frac{\Phi_{21}}{I_1} = \boxed{\mu_0 a^2 \pi \frac{N^2}{l}}$$

$$\therefore \cancel{M = \mu_0 a^2} \quad \boxed{M = \mu_0 a^2 \pi \frac{N^2}{l}}$$

(c)

$L_2 = 40 \times 10^{-3}$ H.

$\therefore M = \frac{L_2}{4} = 10 \times 10^{-3}$ H.

using the magnitudes:

$$\mathcal{E}_1 = M\frac{dI_2}{dt} = (10 \times 10^{-3} H)(2 A s^{-1})$$

$$= \boxed{0.02\ V}$$

F.7.

For a inductor

$P = VI$ & $V = L\frac{dI}{dt}$.

$$U = \int_0^\infty P \cdot dt = \int_0^\infty VI'\,dt.$$

$$= \int_0^\infty L\frac{dI'}{dt} \cdot I' \cdot dt = \int_0^I LI'\,dI'$$

$$= \boxed{\frac{1}{2}LI^2}$$

$\therefore L = \mu_0 \pi a^2 \frac{N^2}{l}$. for a solenoid ~~(a radius~~

(a, N, l has the same meaning as in F.6)

$$\therefore U = \frac{1}{2}\mu_0 \pi a^2 \frac{N^2}{l} I^2 = \frac{1}{2}(\pi a^2 l)(\mu_0 \frac{N}{l} I)^2 (\frac{1}{\mu_0})$$

$\pi a^2 l = \mathcal{V}$ is the volumn inside solenoid.

$B = \mu_0 \frac{N}{l} I$ is the magnetic field inside solenoid

$$\therefore U = \frac{1}{2}(\mathcal{V})(B)^2(\frac{1}{\mu_0}).$$

$\therefore$ Energy per volumn $u = \frac{U}{\mathcal{V}}$

$$u = \boxed{\frac{1}{2}B^2/\mu_0}$$