

EM 3 Ziyun Li

D.0

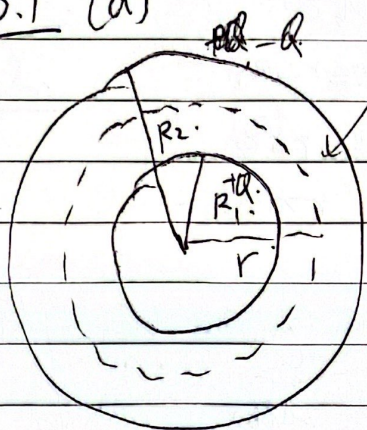
$$C = \frac{Q}{V}$$

Capacitance is the storage of energy through separation of two oppositely polarized charge accumulations.

$$W = \int_0^{V_0} V dQ = \int_0^{V_0} V C dV = \frac{1}{2} C V_0^2$$

$$\therefore W = \frac{1}{2} C V_0^2 = \frac{Q_0^2}{2C} = \frac{1}{2} V_0 Q_0$$

D.1 (a)



Gaussian Surface.

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$E(4\pi r^2) = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \therefore \vec{E} = \frac{Q_{\text{enc}}}{4\pi\epsilon_0 r^2} \hat{r}$$

$$\therefore \vec{E} = \begin{cases} 0 & 0 \leq r < R_1 \\ \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} & R_1 \leq r < R_2 \\ 0 & r > R_2 \end{cases}$$

$$\therefore V = - \int_{R_2}^{R_1} \frac{Q}{4\pi\epsilon_0 r^2} dr$$

$$= - \left[- \frac{Q}{4\pi\epsilon_0 r} \right]_{R_2}^{R_1}$$

$$= \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\therefore C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\epsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)} = \frac{4\pi\epsilon_0}{\frac{1}{R_1} - \frac{1}{R_2}}$$

$$= \boxed{\frac{4\pi\epsilon_0 R_1 R_2}{R_2 - R_1}}$$

(b)

Gauss's Law

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Gaussian Surface

$$\therefore (2\pi r h) E = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$\therefore E = \frac{Q_{\text{enc}}}{2\pi r h \epsilon_0}$$

$$\therefore E = \begin{cases} 0 & 0 \leq r < R_1 \\ \frac{Q}{2\pi r h \epsilon_0} & R_1 \leq r < R_2 \\ 0 & r > R_2 \end{cases}$$

$$V = - \int_{R_2}^{R_1} \frac{Q}{2\pi r h \epsilon_0} dr = - \frac{Q}{2\pi h \epsilon_0} \ln r \Big|_{R_2}^{R_1}$$

$$= \frac{Q}{2\pi h \epsilon_0} \ln \frac{R_2}{R_1}$$

$$\therefore C = \frac{Q}{V} = \frac{Q}{\left[\frac{Q}{2\pi h \epsilon_0} \right] \ln \frac{R_2}{R_1}} = \frac{2\pi h \epsilon_0}{\ln \frac{R_2}{R_1}}$$

(c) as $|R_2 - R_1|$ becomes small. let $R_2 - R_1 = d$
 mod, so $R_2 = R_1 + d$
 for sphere.

$$C = \frac{4\pi \epsilon_0 R_1 R_2}{R_2 - R_1} = \frac{4\pi \epsilon_0 R_1 (R_1 + d)}{d}$$

$$\approx \frac{4\pi R_1^2 \epsilon_0}{d}$$

$A = 4\pi R_1^2$ is the surface area of spheres.

So $C = \frac{\epsilon_0 A}{d}$ as a parallel plate capacitor
 For cylindrical capacitor:

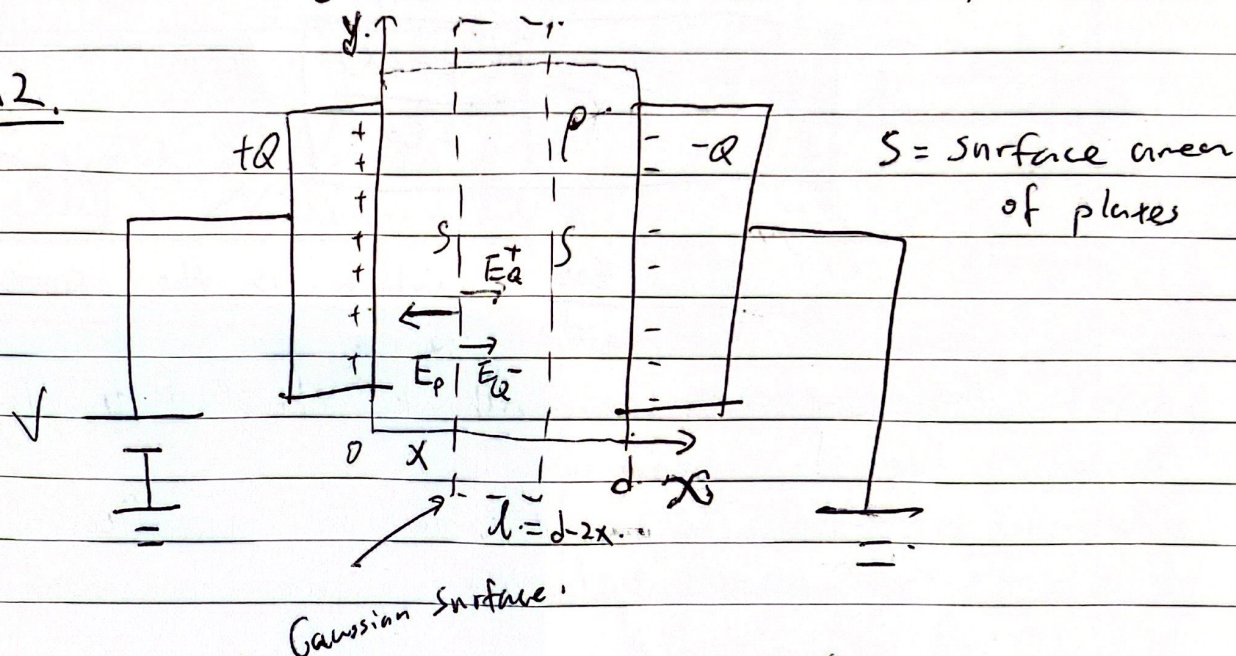
$$C = \frac{2\pi\epsilon_0 l}{\ln(R_2/R_1)} = \frac{2\pi\epsilon_0 l}{\ln(1 + \frac{d}{R_1})}$$

for $x \ll 1$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \dots \approx x$.

$$\therefore C = \frac{2\pi\epsilon_0 l}{\ln(1 + \frac{d}{R_1})} = \frac{2\pi\epsilon_0 l R_1}{d} = \frac{2\pi R_1 l \epsilon_0}{d}$$

$A = 2\pi R_1 l$ is the surface area of the cylinders.
 $\therefore C = \frac{\epsilon_0 A}{d}$ as a parallel plate capacitor.

D.2.



$$\vec{E}_0^+ = \frac{\sigma}{2\epsilon_0} = \frac{Q}{2\epsilon_0 S} \hat{e}_x \quad \vec{E}_0^- = -\frac{\sigma}{2\epsilon_0} = -\frac{Q}{2\epsilon_0 S} \hat{e}_x$$

Applying Gauss's Law:

$$\oint_{\partial V} \vec{E} \cdot d\vec{s} = \frac{Q_v}{\epsilon_0}$$

$$Q_v = P S = P S (d-2x)$$

$$\therefore (E_p)(2S) = P S (d-2x) \quad \therefore \vec{E}_p = \frac{-P(d-2x)}{2\epsilon_0} \hat{e}_x$$

$$\therefore \vec{E} = \vec{E}_0^+ + \vec{E}_0^- + \vec{E}_p = \frac{Q}{\epsilon_0 S} - \frac{P(d-2x)}{2\epsilon_0} \hat{e}_x$$

$$0 - V = - \int_0^d \vec{E} \cdot d\vec{x} \quad \therefore V = \int_0^d \vec{E} \cdot d\vec{x}$$

$$\therefore V = \int_0^d \frac{Q}{\epsilon_0 S} - \frac{\rho(d-2x)}{2\epsilon_0} dx$$

$$= \left[\frac{Qx}{\epsilon_0 S} - \frac{\rho dx}{2\epsilon_0} + \frac{\rho x^2}{2\epsilon_0} \right]_0^d$$

$$= \frac{Qd}{\epsilon_0 S} - \frac{\rho d^2}{2\epsilon_0} + \frac{\rho d^2}{2\epsilon_0} = \frac{Qd}{\epsilon_0 S}$$

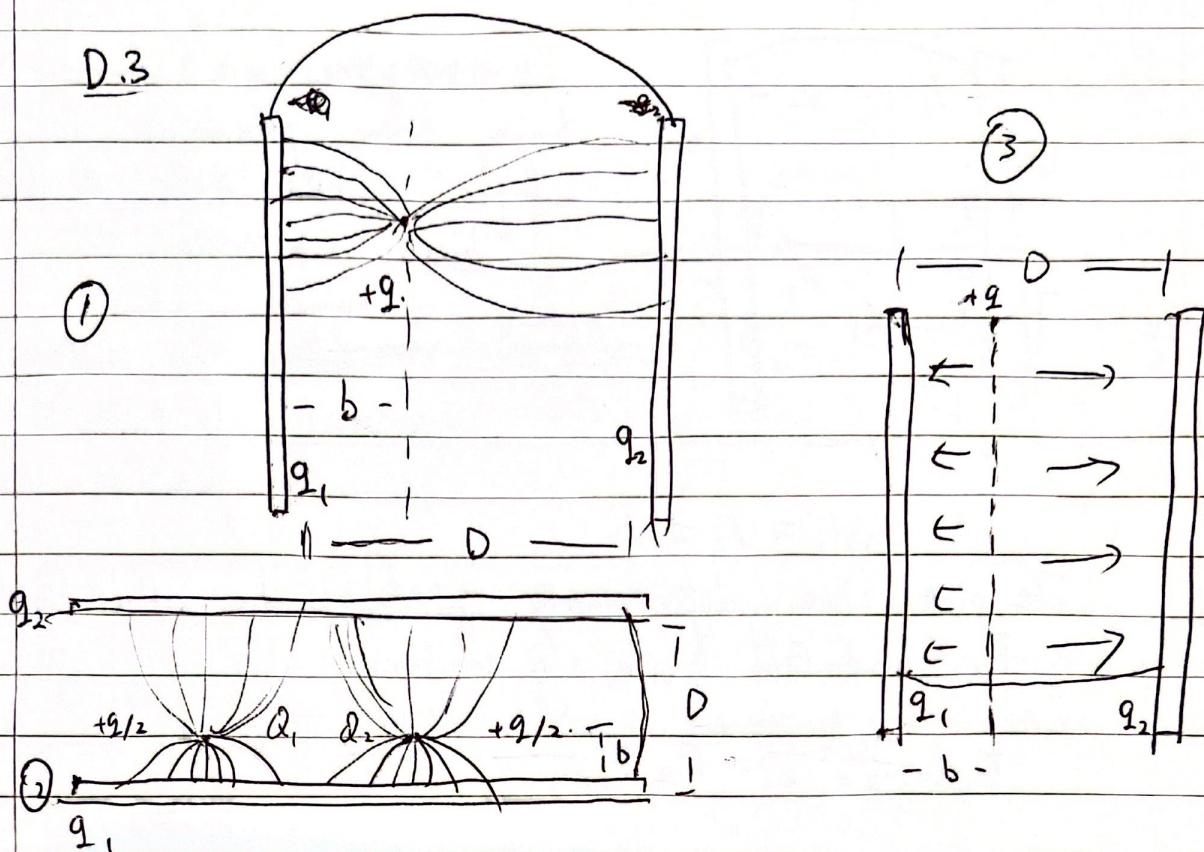
$$\therefore \frac{Qd}{\epsilon_0 S} = V \quad \therefore \frac{Q}{\epsilon_0 S} = \frac{V}{d}$$

$$\therefore \boxed{E = \frac{V}{d} - \frac{\rho(d-2x)}{2\epsilon_0}}$$

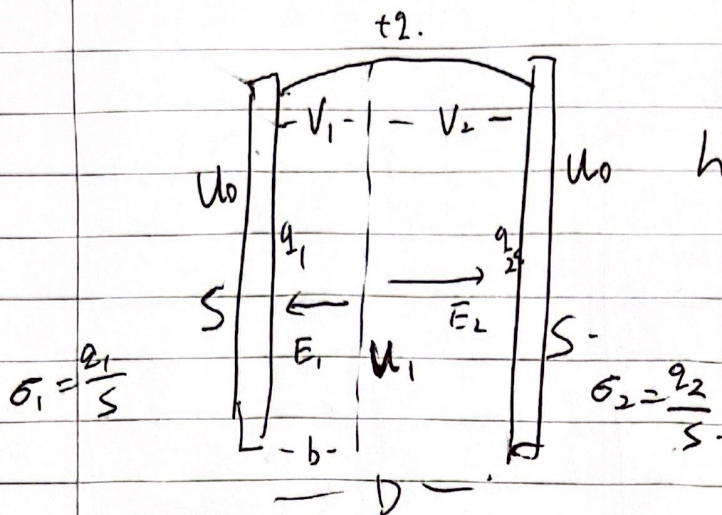
$\therefore C = \frac{Q}{V} = \frac{\epsilon_0 S}{d}$, which is the same as the capacitance without charge distribution.

$\therefore$ The capacitance is unaffected
Only ~~polarized~~ polarized charge distributions can affect capacitance

D.3



As shown in ②, since the two plates are large compare to D , we can treat them as infinite plates. Thus because of symmetry, if two charges of $q_1 = q_2 = +q/2$ are placed a distance b above the bottom plate, the proportion of charges induced on the two plates by q_1 ~~and~~ should be equal to that ~~of~~ by q_2 . Thus if we split the charge $+q$ in ① into two equal charges of $+q/2$, the ~~the~~ distribution of the induced charges on each plates will change, but the proportion of induced charges on each plate will be the same as before. Since ① and ② have the same ~~portion proportion~~ proportion of induced charges on two plates, ~~thus~~ we can split $+q$ in to infinitely ~~many~~ many small charges and ~~put~~ place them to form a sheet of charges, as shown in ③. ① and ③, by our reasoning, should have the same induced charges ~~to~~ proportion.



two plates are connected so have same potential U_0 , let the sheet of t_2 have potential U_1 .

Hence ~~the~~ the difference in potential $V_1 = U_1 - U_0$
 $V_2 = U_1 - U_0$.

$$\therefore V_1 = V_2 = V$$

The plates are approximately infinite so the fields E_1 and E_2 are uniform

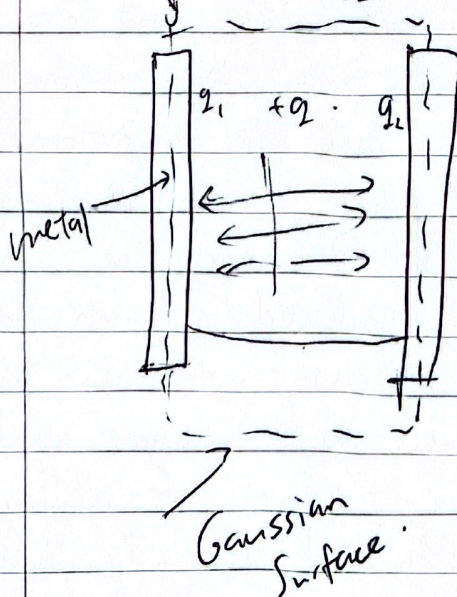
$$\therefore E_1 = \frac{V_1}{b} = \frac{V}{b} \quad E_2 = \frac{V_2}{D-b} = \frac{V}{D-b}$$

let σ_1 and σ_2 be induced surface charge densities

$$\therefore E_1 = \frac{\sigma_1}{\epsilon_0} \quad E_2 = \frac{\sigma_2}{\epsilon_0}$$

$$\therefore \frac{V}{b} = \frac{\sigma_1}{\epsilon_0} = \frac{q_1}{S\epsilon_0} \quad \frac{V}{D-b} = \frac{\sigma_2}{\epsilon_0} = \frac{q_2}{S\epsilon_0}$$

$$\therefore \frac{q_1}{q_2} = \frac{V/b}{V/(D-b)} = \frac{D-b}{b}$$



Apply Gauss's Law to the system.

$$\oint_{\partial V} \vec{E} \cdot d\vec{s} = \frac{Q_V}{\epsilon_0}$$

$\therefore$ Gaussian surface only extends to inside of metal or infinity and field lines are horizontal

$$\therefore \oint \vec{E} \cdot d\vec{s} = 0$$

$$\therefore Q_V = q_1 + q_2 + q = 0$$

$$\therefore q_1 + q_2 = -q$$

$$\therefore \begin{cases} q_1 + q_2 = -q \\ \frac{q_1}{q_2} = \frac{D-b}{b} \end{cases}$$

$$\therefore \boxed{q_1 = -\frac{D-b}{D} q \quad q_2 = -\frac{b}{D} q}$$

D.4

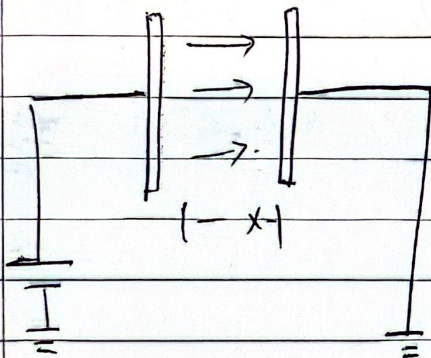
(a)

$$d = 10 \times 10^{-3} \text{ m}$$

$$A = 0.01 \text{ m}^2$$

$$V = 600 \text{ V}$$

The field between capacitor plates ~~here~~ is constant and stores energy with density $U_E = \frac{1}{2} \epsilon_0 E^2$.



$\therefore$ Total potential Energy is

$$U = \frac{1}{2} \epsilon_0 E^2 V = \frac{1}{2} \epsilon_0 E^2 A x.$$

$$E = \frac{V}{x} \quad \therefore U = \frac{1}{2} \epsilon_0 \frac{V^2}{x^2} A x = \frac{1}{2} \epsilon_0 \frac{A V^2}{x}.$$

$$\therefore F = -\frac{\partial U}{\partial x} = -\frac{1}{2} \epsilon_0 A V^2 \frac{\partial}{\partial x} \left(\frac{1}{x} \right) = \frac{1}{2} \epsilon_0 \frac{V^2}{x^2} A.$$

$$\text{when } x = d, \quad F = \frac{1}{2} \epsilon_0 V^2 \frac{A}{d^2}$$

$$\therefore F = \left(\frac{1}{2} \right) (8.854 \times 10^{-12}) (600)^2 (0.01) / (10 \times 10^{-3})^2 \\ = \underline{1.59 \times 10^{-4} \text{ N}}$$

(b) (i)

~~$$W = U_2 - U_1 = \frac{1}{2} \epsilon_0 A V^2 \left(\frac{1}{x_2} - \frac{1}{x_1} \right)$$~~

$$W = -\Delta U = U_1 - U_2 = \frac{1}{2} \epsilon_0 A V^2 \left(\frac{1}{x_1} - \frac{1}{x_2} \right)$$

$$x_1 = d \quad x_1 = 10 \times 10^{-3} \text{ m} \quad x_2 = 20 \times 10^{-3} \text{ m}.$$

$$\therefore W = \frac{1}{2} (8.854 \times 10^{-12}) (600)^2 (0.01) \left(\frac{1}{10 \times 10^{-3}} - \frac{1}{20 \times 10^{-3}} \right)$$

$$= \boxed{7.97 \times 10^{-7} \text{ J}}$$

(ii) If plates are isolated before separation.

let $\sigma = \frac{Q}{A}$ be the surface charge density.

$$\therefore E = \frac{\sigma}{\epsilon_0} \quad U(x) = \frac{1}{2} \epsilon_0 E^2 A x = \frac{1}{2} \frac{\sigma^2}{\epsilon_0} A x = \frac{1}{2} \frac{Q^2}{\epsilon_0 A^2} A x = \frac{1}{2} \frac{Q^2}{\epsilon_0 A} x$$

$$\therefore F = - \frac{\partial U}{\partial x} = - \frac{1}{2} \frac{Q^2}{\epsilon_0 A}$$

$$\therefore \text{Work done } W = - \int_{x_1}^{x_2} F \cdot dx = \frac{1}{2} \frac{Q^2}{\epsilon_0 A} (x_2 - x_1)$$

$$\therefore \cancel{W = \frac{1}{2} \epsilon_0 E^2 A x} \quad Q = \sigma A = \epsilon_0 E A = \frac{\epsilon_0 A}{d} V$$

$$\therefore W = \frac{1}{2} \frac{\epsilon_0^2 A^2}{d^2} V^2 \frac{1}{\epsilon_0 A} (x_2 - x_1)$$

$$\therefore W = \frac{1}{2} \epsilon_0 A \frac{V^2}{d^2} (x_2 - x_1)$$

$$= \frac{1}{2} (8.854 \times 10^{-12}) (0.01) (600)^2 \cdot \left(\frac{20 \text{ mm}}{10 \text{ mm}} \right) \times \left(\frac{1}{(0.01)^2} \right) (0.02 - 0.01)$$

$$= \boxed{1.59 \times 10^{-6} \text{ J}}$$

(c) Energy stored before:

$$U_1 = \frac{1}{2} \epsilon_0 A V^2 / x_1 = 1.59 \times 10^{-6} \text{ J}$$

$$U_2 = \frac{1}{2} \epsilon_0 A V^2 / x_2 = 7.97 \times 10^{-7} \text{ J}$$

$$\therefore \frac{U_2}{U_1} = \frac{1}{2}$$

The work done on the system goes into the battery and charges the battery up.

E.0

Biot-Savart Law:

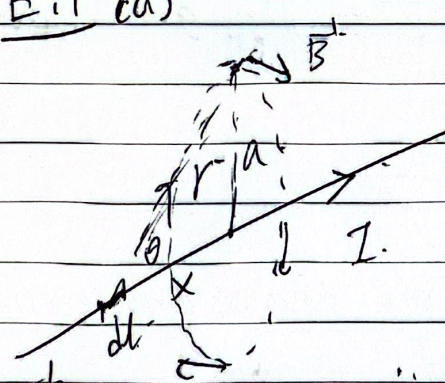
$$d\vec{B} = \frac{\mu_0 I}{4\pi r^2} d\vec{l} \times \hat{r}$$

Ampere's Law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \quad (I \text{ is current penetrating the loop})$$

Ampere ~~Law~~ is used for symmetrical situations where it is easy to find a ~~current~~ loop has constant magnetic field magnitude and is penetrated by a known current, whereas Biot-Savart is ~~used~~ used for less symmetrical situations.

E.1 (a)



$$d\vec{B} = \frac{\mu_0 I}{4\pi r^2} d\vec{l} \times \hat{r}$$

$$dB = \frac{\mu_0 I}{4\pi r^2} dl \sin\theta$$

$$\sin\theta = \frac{a}{\sqrt{x^2 + a^2}} \quad r = \sqrt{x^2 + a^2} \quad dl = dx$$

$$\therefore dB = \frac{\mu_0 I}{4\pi} \frac{a dx}{(x^2 + a^2)^{3/2}}$$

$$= \frac{\mu_0 I a}{4\pi} \int_{-b}^b \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{\mu_0 I a}{4\pi} \int_{-b}^b \frac{dx}{x^3 \left(1 + \frac{a^2}{x^2}\right)^{3/2}}$$

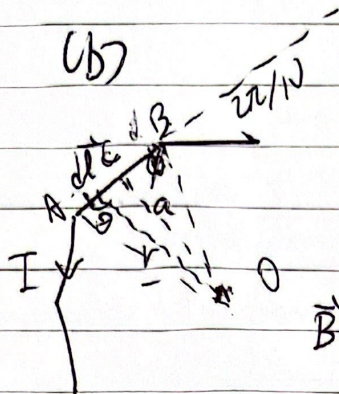
$$= \frac{\mu_0 I a}{4\pi} \int_{x=-b}^{x=b} \frac{-x^3 du}{2a^2 x^3 u^{3/2}} = \frac{\mu_0 I}{8\pi a} \int_{x=-b}^{x=b} \frac{du}{u^{3/2}}$$

$$= \frac{\mu_0 I}{8\pi a} \left[-\frac{1}{u^{1/2}} \right]_{x=-b}^{x=b} = \frac{\mu_0 I}{8\pi a} \left[\frac{1}{\sqrt{1 + \frac{a^2}{x^2}}} \right]_{-b}^b$$

$$= \frac{\mu_0 I}{8\pi a} \left[\frac{x}{\sqrt{x^2 + a^2}} \right]_{-b}^b = \frac{\mu_0 I}{4\pi a} \frac{b}{\sqrt{a^2 + b^2}} = \frac{\mu_0 I b}{2\pi a \sqrt{a^2 + b^2}}$$

$u = 1 + \frac{a^2}{x^2}$
 $\frac{du}{dx} = -\frac{2a^2}{x^3}$
 $\therefore dx = \frac{-x^3}{2a^2} du$

$$\therefore H = \frac{1}{\mu_0} B = \left[\frac{Ib}{2\pi a \sqrt{a^2 + b^2}} \right]$$



$$\angle OBC = \phi = \frac{1}{2} \left(\pi - \frac{2\pi}{N} \right)$$

$$= \frac{\pi}{2} - \frac{\pi}{N}$$

all wires produce B-field to the direction out of page

$\therefore$ For one wire. $B_1 = \frac{B}{N}$

Define $B_2 = \frac{B_1}{2} = \frac{1}{2}$ of B field produced by half of the wire.

$$B_2 = \int_{\theta=\phi}^{\theta=\pi/2} \frac{\mu_0 I}{4\pi r^2} dl \sin \theta \quad dl = dx$$

$$\sin \theta = \frac{a}{r} \quad \therefore r = \frac{a}{\sin \theta}$$

$$\tan \theta = \frac{a}{-x} \quad \text{(- sign indicates direction)}$$

$$x = -a \cot \theta$$

$$x = -a \cot \theta$$

$$\therefore \frac{dx}{d\theta} = -(-\csc^2 \theta) a = \frac{a}{\sin^2 \theta}$$

$$\therefore dx = \frac{a d\theta}{\sin^2 \theta}$$

$$\therefore B_2 = \int_{\phi}^{\pi/2} \frac{\mu_0 I}{4\pi r^2} (\sin \theta) \left(\frac{\sin^2 \theta}{a^2} \right) \left(\frac{a}{\sin^2 \theta} \right) d\theta$$

$$= \frac{\mu_0 I}{4\pi a} \int_{\phi}^{\pi/2} \sin \theta d\theta = \frac{\mu_0 I}{4\pi a} [-\cos \theta]_{\phi}^{\pi/2}$$

$$= \frac{\mu_0 I}{4\pi a} \left[1 - \cos \left(\frac{\pi}{2} - \frac{\pi}{N} \right) \right] = \frac{\mu_0 I}{4\pi a} \left[1 - \sin \left(\frac{\pi}{N} \right) \right]$$

$$= \frac{N_0 I}{4\pi a} [0 - (-\cos(\frac{\pi}{2} - \frac{\pi}{N}))]$$

$$= \frac{N_0 I}{4\pi a} \sin(\frac{\pi}{N})$$

$$\therefore B_1 = 2B_2 = \frac{N_0 I}{2\pi a} \sin(\frac{\pi}{N})$$

$$B = NB_1 = \frac{N N_0 I \sin(\pi/N)}{2\pi a}$$

$$\therefore H = \frac{1}{\mu_0} B = \frac{N I \sin(\pi/N)}{2\pi a}$$

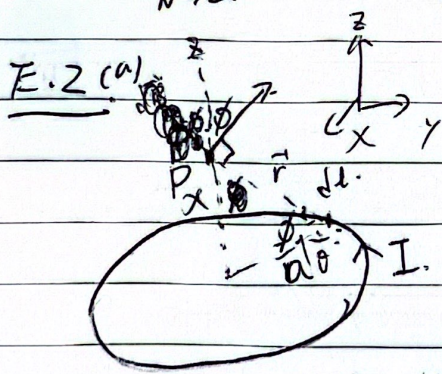
$$\boxed{\frac{N I \sin(\pi/N)}{2\pi a}}$$

$$\lim_{N \rightarrow \infty} H =$$

when $N \rightarrow \infty$

$$\frac{\pi}{N} \rightarrow 0 \quad \therefore \sin(\pi/N) \rightarrow \frac{\pi}{N}$$

$$\therefore \lim_{N \rightarrow \infty} H = \frac{N I (\pi/N)}{2\pi a} = \boxed{\frac{I}{2a}}$$



$$dB = \frac{\mu_0 I}{4\pi r^2} dl \times \hat{r} \quad \therefore dl \perp \hat{r} \quad \therefore dl \times \hat{r} = dl$$

$\therefore dB$'s, However, points in different directions. Their effects

in the x, y directions cancel

so only the z direction has magnetic field.

$$\therefore B_z = \int dB \cos \phi \quad (\cos \phi = \frac{a}{\sqrt{a^2 + z^2}}) \quad dl = a d\theta$$

$$\therefore B_z = \int_0^{2\pi} dB \cos \phi = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a (a d\theta)}{(a^2 + z^2)^{3/2}}$$

$$= \frac{\mu_0 I a^2}{(a^2 + z^2)^{3/2}}$$

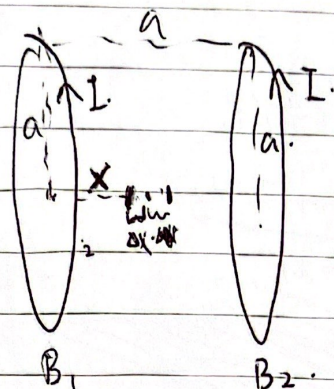
$\therefore$ Coil has N turns.

$$\therefore B = NB_z = \frac{N \mu_0 I a^2}{(a^2 + z^2)^{3/2}}$$

$$\therefore H = \frac{1}{\mu_0} B =$$

$$\frac{N I a^2}{2(a^2 + x^2)^{3/2}}$$

(b)



Let the displacement from the middle be Δx .

$$B_1 = \frac{\mu_0 N I a^2}{2(a^2 + (\frac{a}{2} - \Delta x)^2)^{3/2}}$$

$$B_2 = \frac{\mu_0 N I a^2}{2(a^2 + (\frac{a}{2} + \Delta x)^2)^{3/2}}$$

Let $\frac{\mu_0 N I a^2}{2} = k$ which is a constant

$$B = B_1 + B_2 = k \left[(a^2 + \frac{a^2}{4} - a\Delta x + \Delta x^2)^{-3/2} + (a^2 + \frac{a^2}{4} + a\Delta x + \Delta x^2)^{-3/2} \right]$$

$$\therefore B = k \left[\left(\frac{5}{4}a^2 - a\Delta x + \Delta x^2 \right)^{-3/2} + \left(\frac{5}{4}a^2 + a\Delta x + \Delta x^2 \right)^{-3/2} \right]$$

$$= \frac{8k}{5\sqrt{5}a^3} \left[\left(1 - \frac{4\Delta x}{5a} \right)^{-3/2} + \left(1 + \frac{4\Delta x}{5a} \right)^{-3/2} \right]$$

$$\text{Let } \frac{8k}{5\sqrt{5}a^3} = C, \quad \frac{4\Delta x}{5a} = d$$

$$B = C \left[(1-d)^{-3/2} + (1+d)^{-3/2} \right]$$

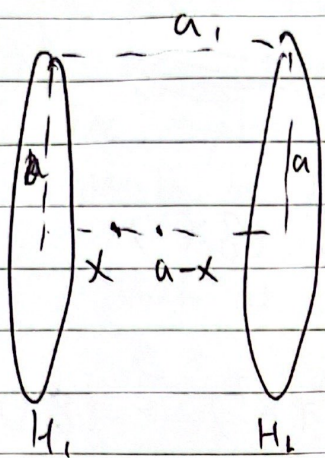
$$B = k \left[\left(\frac{5}{4}a^2 - a\Delta x + \Delta x^2 \right)^{-3/2} + \left(\frac{5}{4}a^2 + a\Delta x + \Delta x^2 \right)^{-3/2} \right]$$

$$= \frac{8k}{5\sqrt{5}a^3} \left[\left(1 - \frac{4\Delta x}{5a} + \frac{4\Delta x^2}{5a^2} \right)^{-3/2} + \left(1 + \frac{4\Delta x}{5a} + \frac{4\Delta x^2}{5a^2} \right)^{-3/2} \right]$$

$$\text{Let } \frac{8k}{5\sqrt{5}a^3} = C, \quad \therefore \Delta x \ll a$$

$$B = C \left[\left(1 - \frac{3}{2} \left(\frac{4\Delta x}{5a} + \frac{4\Delta x^2}{5a^2} \right) + \frac{15}{4} \left(\frac{4\Delta x}{5a} + \frac{4\Delta x^2}{5a^2} \right)^2 \right) \right]$$

$$- \frac{105}{8(3!)} \left(\frac{4\Delta x}{5a} + \right)$$



$$H = H_1 + H_2 = \frac{NIa^2}{2(a^2+x^2)^{3/2}} + \frac{NIa^2}{2(a^2+(a-x)^2)^{3/2}}$$

$$\text{Let } \frac{NIA^2}{2} = k.$$

~~$$H = k \left[(a^2+x^2)^{-3/2} + (a^2+(a-x)^2)^{-3/2} \right]$$~~

$$H = k \left[(a^2+x^2)^{-3/2} + (a^2+(a-x)^2)^{-3/2} \right]$$

$$\therefore \frac{\partial H}{\partial x} = k \left[-\frac{3}{2} (a^2+x^2)^{-5/2} (2x) - \frac{3}{2} (a^2+(a-x)^2)^{-5/2} (2(x-a)) \right]$$

$$\therefore \frac{\partial H}{\partial x} \Big|_{x=\frac{a}{2}} = k \left[-\frac{3}{2} \left(\frac{5}{4} a^2 \right)^{-5/2} (a) + \frac{3}{2} \left(\frac{5}{4} a^2 \right)^{-5/2} (a) \right] = 0$$

$$\frac{\partial^2 H}{\partial x^2} = -\frac{3}{2} k \left[-\frac{5}{2} (a^2+x^2)^{-7/2} (2x)(2x) + (a^2+x^2)^{-7/2} (2) \right.$$

$$\left. -\frac{5}{2} (a^2+(a-x)^2)^{-7/2} (2(x-a))(2(x-a)) + (a^2+(a-x)^2)^{-7/2} (2) \right]$$

$$\frac{\partial^2 H}{\partial x^2} \Big|_{x=\frac{a}{2}} = -\frac{3}{2} k \left[-\frac{5}{2} \left(\frac{5}{4} a^2 \right)^{-7/2} (a^2) + 2 \left(\frac{5}{4} a^2 \right)^{-7/2} \right] \times 2$$

$$= -3k \left[\underbrace{-\frac{5}{2} \left(\frac{5}{4} \right)^{-7/2} + 2 \left(\frac{5}{4} \right)^{-7/2}}_{=0} \right] a^{-3/2}$$

$$= 0$$

$$\frac{\partial^3 H}{\partial x^3} = -\frac{3k}{2} \left\{ \begin{aligned} &\frac{35}{4} (a^2 + x^2)^{-\frac{9}{2}} (2x) (8x^3) \\ &- \frac{5}{2} (a^2 + x^2)^{-\frac{7}{2}} (8x) \end{aligned} \right.$$

$$\begin{aligned} &- 5(a^2 + x^2)(2x) + \frac{35}{4} (a^2 + x^2)^{-\frac{9}{2}} (8(x-a)^3) \\ &- \frac{5}{2} (a^2 + (a-x)^2)^{-\frac{7}{2}} (8(x-a)) \\ &- 5(a^2 + (a-x)^2)^{-\frac{7}{2}} (2(x-a)) \end{aligned} \left. \vphantom{\frac{35}{4}} \right\}$$

$$\left. \frac{\partial^3 H}{\partial x^3} \right|_{x=\frac{a}{2}} = -\frac{3k}{2} \left\{ \begin{aligned} &\frac{35}{4} \left(\frac{5}{4}a^2\right)^{-\frac{9}{2}} (a^3) - \frac{35}{4} \left(\frac{5}{4}a^2\right)^{-\frac{9}{2}} (a^3) \\ &- \frac{5}{2} \left(\frac{5}{4}a^2\right)^{-\frac{7}{2}} (4a) + \frac{5}{2} \left(\frac{5}{4}a^2\right)^{-\frac{7}{2}} (4a) \\ &- 5 \left(\frac{5}{4}a^2\right) (a) + 5 \left(\frac{5}{4}a^2\right) (a) \end{aligned} \right\}$$

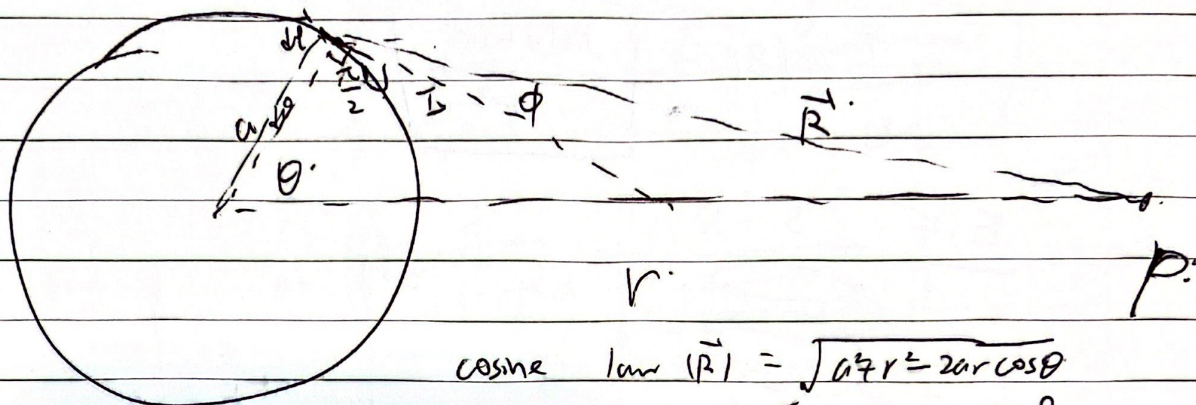
$$= 0.$$

$$= 0.$$

$$\therefore \frac{\partial H}{\partial x}, \frac{\partial^2 H}{\partial x^2}, \frac{\partial^3 H}{\partial x^3} \text{ all are equal to } 0$$

The Helmholtz coil can be used to create locally constant magnetic fields. because the Taylor expansion of the B-field about $x = \frac{a}{2}$ would be $B(x) = B(\frac{a}{2}) + (\frac{\partial B}{\partial x} / 1!) B'(x - \frac{a}{2}) + \dots$ which is nearly constant around $x = \frac{a}{2}$.

E.3



cosine law $|\vec{R}| = \sqrt{a^2 + r^2 - 2ar \cos \theta}$
 sine law: $\frac{\sin(\phi + \frac{\pi}{2})}{r} = \frac{\sin \theta}{\sqrt{a^2 + r^2 - 2ar \cos \theta}}$

$\therefore \cos \phi = \frac{r \sin \theta}{\sqrt{a^2 + r^2 - 2ar \cos \theta}}$

$\sin^2 \phi = 1 - \cos^2 \phi = \frac{a^2 + r^2 - 2ar \cos \theta - r^2 \sin^2 \theta}{a^2 + r^2 - 2ar \cos \theta} = \frac{(r \cos \theta - a)^2}{a^2 + r^2 - 2ar \cos \theta}$

$\therefore \sin \phi = \frac{r \cos \theta - a}{\sqrt{a^2 + r^2 - 2ar \cos \theta}}$ $\vec{dl} \times \hat{R} = \sin \phi$
 $dl = a d\theta$

$\therefore d\vec{B}_1 = \frac{\mu_0 I}{4\pi R^2} \vec{dl} \times \hat{R}$ $\therefore B_1 = \frac{\mu_0 I a^2}{4\pi} \int_0^{2\pi} \frac{(r \cos \theta - a)}{(a^2 + r^2 - 2ar \cos \theta)^{3/2}} d\theta$

for $r \gg a$.

$(a^2 + r^2 - 2ar \cos \theta)^{-3/2} \approx (r^2 - 2ar \cos \theta)^{-3/2}$
 $= \frac{1}{r^3} (1 - \frac{2a}{r} \cos \theta)^{-3/2} \approx \frac{1}{r^3} (1 + \frac{3a}{r} \cos \theta)$

$\therefore B_1 \approx \frac{\mu_0 I a}{4\pi r^3} \int_0^{2\pi} (r \cos \theta - a) (1 + \frac{3a}{r} \cos \theta) d\theta$

$$= \frac{\mu_0 I a}{4\pi r^3} \int_0^{2\pi} r \cos \theta - a + 3a \cos^2 \theta - \frac{3a^3}{r} \cos \theta d\theta$$

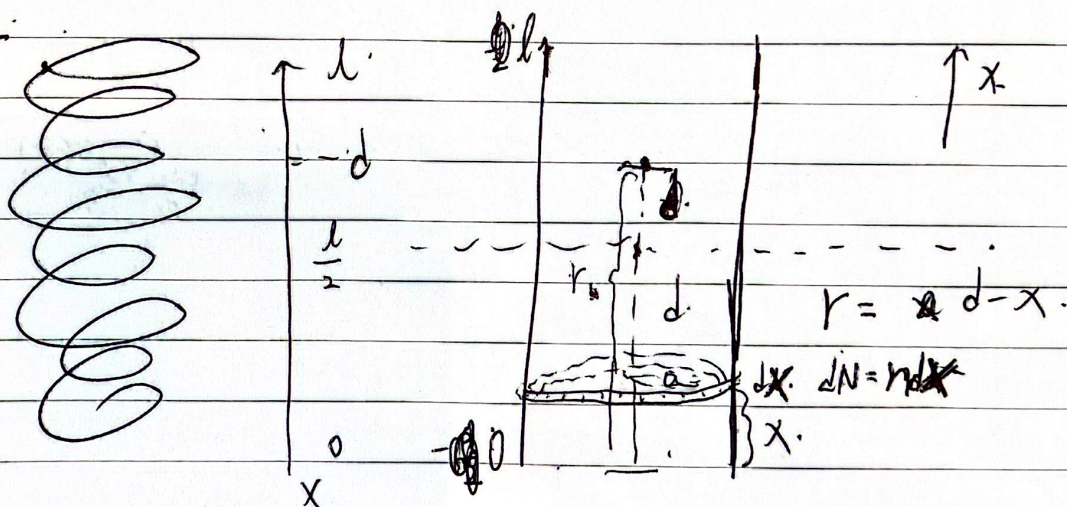
$$\therefore \int_0^{2\pi} \cos^2 \theta d\theta = \pi \quad \int_0^{2\pi} \cos \theta d\theta = 0$$

$$\therefore B_1 = \frac{\mu_0 I a}{4\pi r^3} \left[-2\pi a + 3\pi a \right]$$

$$= \frac{\mu_0 I a}{4\pi r^3} (\pi a) = \frac{\mu_0 I a^2}{4r^3}$$

$$\therefore B = N B_1 = \boxed{\frac{\mu_0 N I a^2}{4r^3}}$$

E.4.



Using the result of E.2

$$dB = \frac{dN I a^2}{2(a^2 + r^2)^{3/2}} = \frac{dN I a^2}{2(a^2 + (d-x)^2)^{3/2}}$$

$$= \frac{n I a^2 dx}{2(a^2 + (d-x)^2)^{3/2}}$$

$$\therefore B_0 \quad B(d) = \int_0^l \frac{n I a^2}{2} \frac{dx}{(a^2 + (d-x)^2)^{3/2}}$$

substitute $u = d - x$ $du = -dx$ $dx = -du$

$$\text{let } \int_0^l \frac{dx}{(a^2 + (d-x)^2)^{3/2}} = \int_d^{d-l} \frac{-du}{(a^2 + u^2)^{3/2}}$$

$$= \int_{d-l}^d \frac{du}{(a^2 + u^2)^{3/2}} = \left[\frac{u/a^2}{(a^2 + u^2)^{1/2}} \right]_{d-l}^d$$

$$= \frac{1}{a^2} \left[\frac{d}{\sqrt{a^2 + d^2}} - \frac{d-l}{\sqrt{a^2 + (d-l)^2}} \right]$$

$$= \frac{1}{a^2} \left[\frac{d}{\sqrt{a^2 + d^2}} + \frac{1-d}{\sqrt{a^2 + (1-d)^2}} \right]$$

$$\therefore B(d) = \frac{\mu_0 n I}{2} \left[\frac{d}{\sqrt{a^2 + d^2}} + \frac{1-d}{\sqrt{a^2 + (1-d)^2}} \right]$$

$$B(d = \frac{1}{2}) = \frac{\mu_0 n I}{2} \left[\frac{1/2}{\sqrt{a^2 + 1/4}} \times 2 \right]$$

$$= \frac{\mu_0 n I}{2} \frac{1}{\sqrt{a^2 + 1/4}}$$

$$B(d=0) = B(d=1) = \frac{\mu_0 n I}{2} \frac{1}{\sqrt{a^2 + 1^2}}$$

We are given that $B(d = \frac{1}{2}) = \sqrt{2} [B(d=0)]$

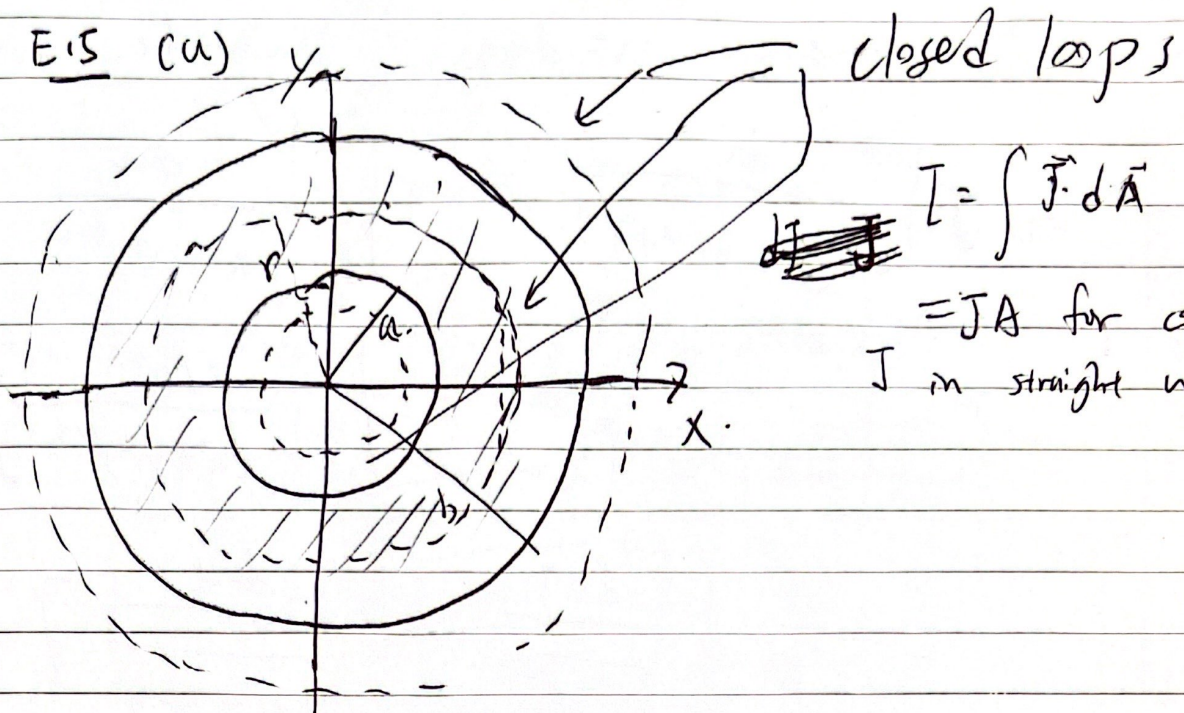
$$\therefore \frac{\sqrt{a^2 + 1^2}}{\sqrt{a^2 + \frac{1}{4}}} = \sqrt{2} \quad \therefore a^2 + 1^2 = 2(a^2 + \frac{1}{4})$$

$$a^2 + 1^2 = 2a^2 + \frac{1}{2} 1^2$$

$$\therefore \frac{1}{2} 1^2 = a^2$$

$$\therefore \frac{1^2}{a^2} = 2$$

$$\therefore \boxed{\frac{1}{a} = \sqrt{2}}$$



(i) $r < a$.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{A} = \mu_0 \int \vec{j} \cdot d\vec{A}$$

$$\therefore \int \vec{J} \cdot d\vec{A} = 0 \because \vec{B} = 0 \therefore \underline{H = 0}$$

(ii) ~~a < r < b~~ $a < r < b$

~~$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{A}$$~~

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{j} \cdot d\vec{A}$$

$$\vec{j} = \vec{J} \therefore \int \vec{j} \cdot d\vec{A} = J(\pi(r^2 - a^2))$$

$$\therefore B(2\pi r) = \mu_0 J(\pi(r^2 - a^2)).$$

$$\therefore B = \frac{\mu_0 J \pi(r^2 - a^2)}{2\pi r} = \frac{\mu_0 J (r^2 - a^2)}{2r}$$

$$\therefore \boxed{H = \frac{J(r^2 - a^2)}{2r}}$$

(iii)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{j} \cdot d\vec{A}$$

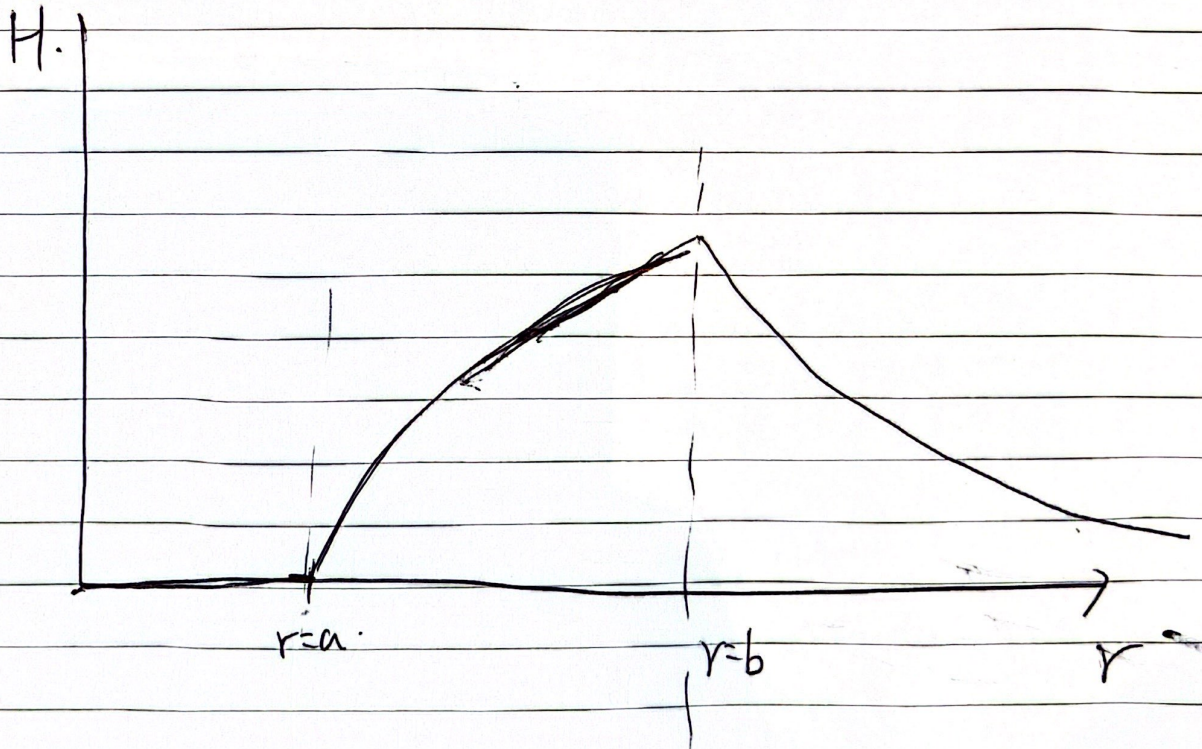
$$\int \vec{j} \cdot d\vec{A} = J(\pi(b^2 - a^2))$$

$$\therefore B(2\pi r) = \mu_0 J(\pi(b^2 - a^2))$$

$$\therefore B = \frac{\mu_0 J(b^2 - a^2)}{2r}$$

$$H = \frac{J(b^2 - a^2)}{2r}$$

(b)



Addition to E.3

The field density of a magnetic dipole is given by

$\vec{B} = \vec{\nabla} \times \vec{A}$ where $\vec{A}$ is the ~~vector~~ magnetic vector potential and has the expression.

$$\vec{A} = \frac{\mu_0}{4\pi r^2} \frac{\vec{m} \times \vec{r}}{r} \quad \text{where } \vec{m} \text{ is the}$$

magnetic dipole moment that has magnitude $m = IS$ ($S = \text{area of loop}$) and direction ~~normal~~ to normal to the loop. In our case of a circular loop, $\vec{m} = \pi a^2 I \hat{n}$

$\vec{B}$ -field produced by a dipole is

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \left(\frac{3\vec{r}(\vec{m} \cdot \vec{r})}{r^5} - \frac{\vec{m}}{r^3} \right)$$

$\therefore$ the loop and the point P where the B -field is calculated is on the same plane.

$\therefore \vec{m}$ is normal to $\vec{r} \quad \therefore \vec{m} \cdot \vec{r} = 0$

$$\therefore \vec{B} = -\frac{\mu_0 \vec{m}}{4\pi r^3} \quad \text{Taking the magnitude}$$

$$B = \frac{\mu_0 m}{4\pi r^3}$$

$\therefore m = \pi a^2 I$ and there are N loops

$$\therefore \boxed{B = \frac{\mu_0 N I a^2}{4r^3}}$$

which is the same as that produced by the coil.