EM 2

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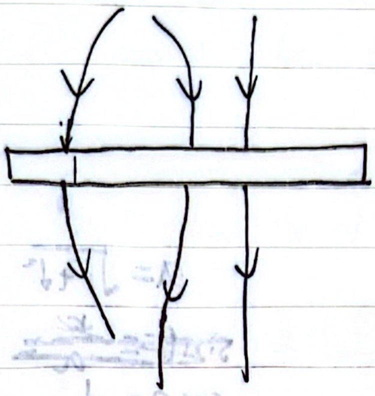
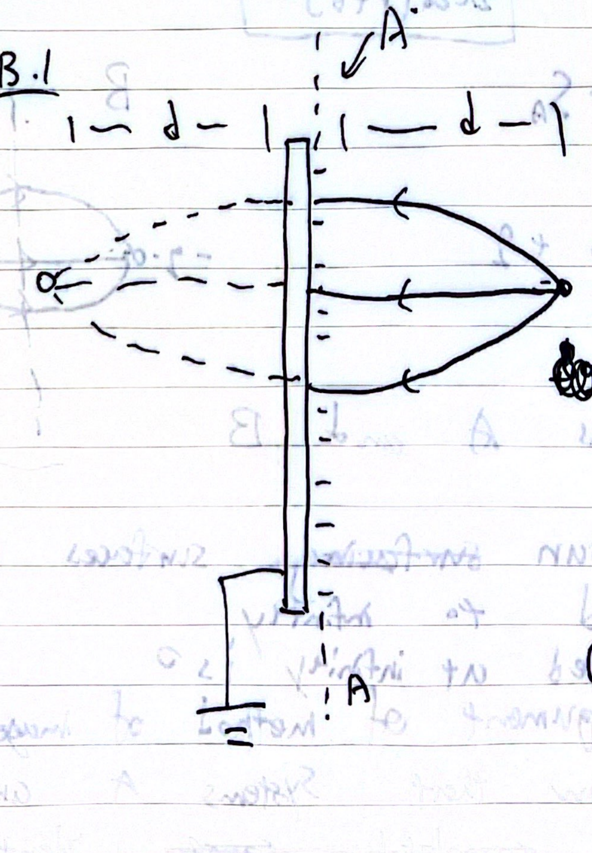
B.0



The electric field lines ~~is~~ are normal to the conductor near its surface because if there is a tangential component than the charge on the surface of the conductor will experience force and move.

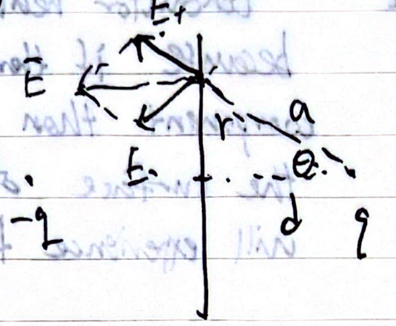
B.1



Using method of images we ~~re~~replace the ~~[illegible]~~ metal plate with a imaginary charge a distance d away from the plate in the other direction. -q. creates the same boundary condition ~~[illegible]~~ as ~~the~~ ~~[illegible]~~ the metal plate does at surface A. (The boundary condition at A is $V_A = 0$).

(a) $\therefore \boxed{F = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d^2}}$

(b)



$a = \sqrt{r^2+d^2}$

$\cancel{\sin\theta = \frac{r}{a}}$

$\cos\theta = \frac{d}{a} = \frac{d}{\sqrt{r^2+d^2}}$

$\cancel{E = E_+\sin\theta + E_-\sin\theta}$ $E = E_+\cos\theta + E_-\cos\theta$

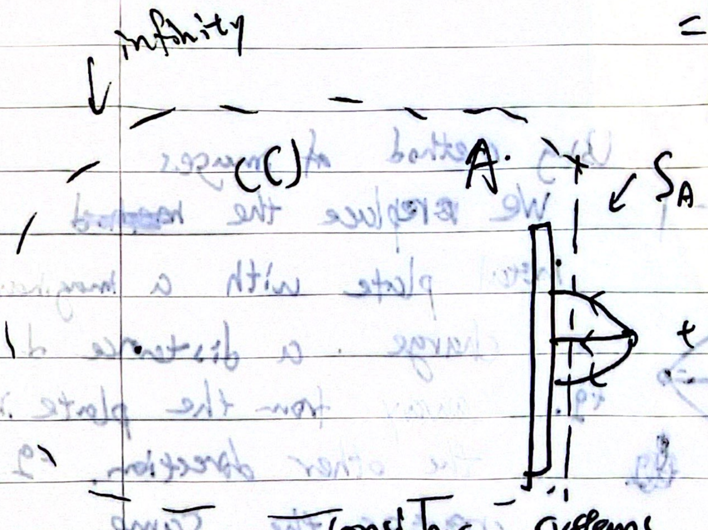
$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{a^2}\right)\times 2\cos\theta$

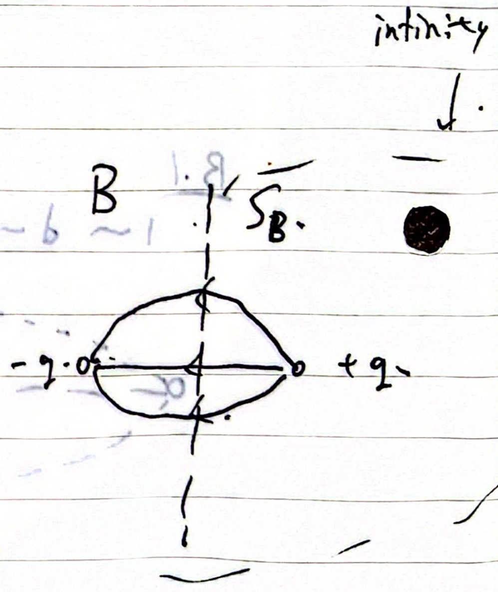
$\cancel{E_+ =}$

$= \frac{2qd}{4\pi\varepsilon_0 (r^2+d^2)^{3/2}}$

$= \boxed{\frac{qd}{2\pi\varepsilon_0 (r^2+d^2)^{3/2}}}$

(c)





Consider systems A and B

~~Making~~ Consider Gaussian ~~surfacing~~ surfaces S_A and S_B, both of which extend to infinity

The electric fied at infinity is 0

And By the argument of method of images we know that Systems A and B have ~~completely same~~ identical electric fields, Thus the electric flux through S_A and S_B must be the same

~~By Gaussi Gaussian Gaussi~~

By Gauss's Law

$$\oiint E \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$

$\because$ Total flux through the ~~enclose~~ surfaces are the same for A and B

$\therefore$ system A and B must have same enclosed charge.

$\therefore$ The induced charge on the plate is the same as the image charge, which is $\boxed{-q}$.

(d)

The work done in moving the charge two infinity ~~is~~ in system A is equivalent to doing the same work to both charges to ~~move~~ move them to infinity in system B. Thus let work done be W, then

$$2W = |U_B| = \cancel{\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q^2}{d^2}}$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q^2}{2d}.$$

$$\therefore W = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q^2}{4d}\right) = \boxed{\frac{q^2}{16\pi\varepsilon_0 d}}$$

ii, The minimum energy

$$V_0 = \frac{Q^2}{16\pi\varepsilon_0 d}$$

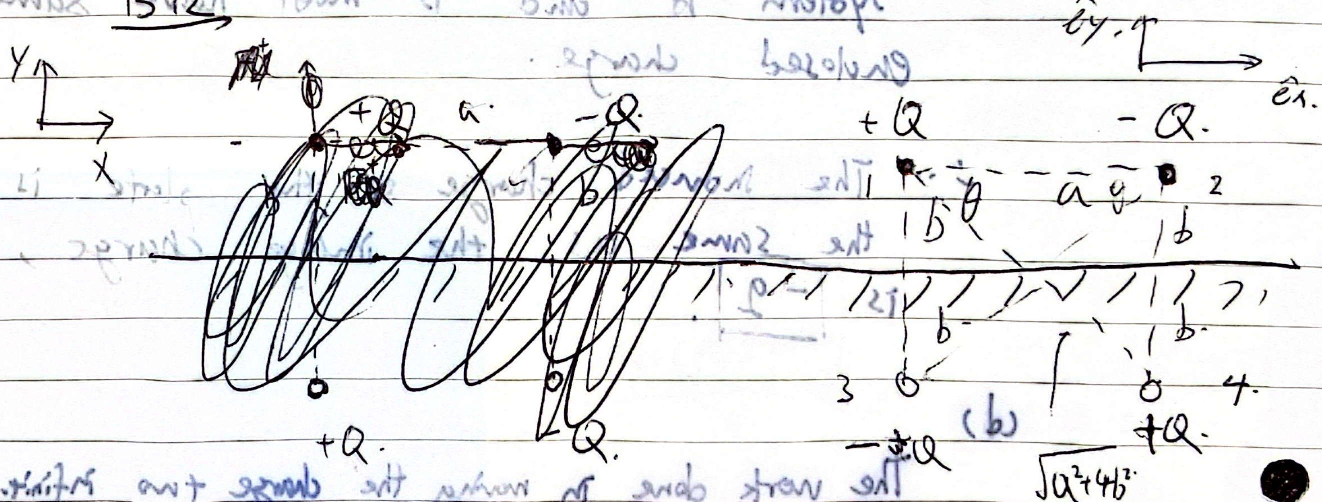
given that $d = 0.1\,nm$, $|q| = |e| = 1.6\times10^{-19}\,C$.

$\varepsilon_0 = 8.854\times10^{-12}\,F\,m^{-1}$

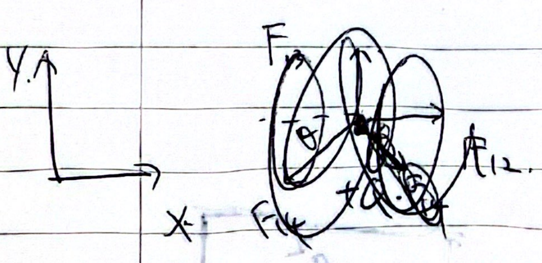
$$\therefore V_0 = \frac{(1.6\times10^{-19}\,C)^2}{16\pi(8.854\times10^{-12}\,F\,m^{-1})(0.1\times10^{-9}\,m)} = 5.75\times10^{-19}\,J$$

$$= 3.60\,eV$$

B2)



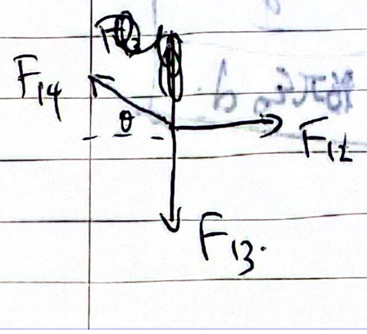
The induced charges on the conducting sheet can be replaced by two image charges on the other side of the sheet.



For +Q, $\cos\theta = \frac{a}{\sqrt{a^2+4b^2}}$, $\sin\theta = \frac{2b}{\sqrt{a^2+4b^2}}$

$$F_x = F_{12} - F_{14}\cos\theta$$

$$= \frac{Q^2}{4\pi\varepsilon_0 a^2} - \frac{Q^2}{4\pi\varepsilon_0 (\sqrt{a^2+4b^2})^2}\cos\theta$$



$$= \frac{Q^2}{4\pi\varepsilon_0 a^2} - \frac{aQ^2}{4\pi\varepsilon_0 (a^2+4b^2)^{3/2}} = \boxed{\frac{Q^2}{4\pi\varepsilon_0}\left(\frac{1}{a^2} - \frac{a}{(a^2+4b^2)^{3/2}}\right)}$$

$$F_y^+ = (F_{14}\sin\theta) - F_{13}$$

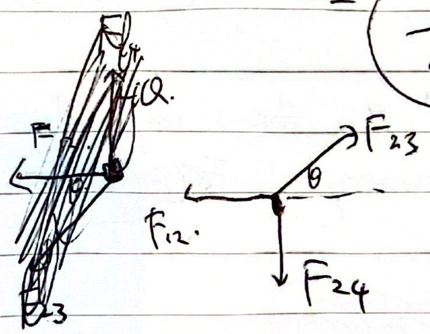
$$= -\left(\frac{Q^2}{4\pi\varepsilon_0(2b)^2} - \frac{Q^2}{4\pi\varepsilon_0(a^2+4b^2)}\left(\frac{2b}{\sqrt{a^2+4b^2}}\right)\right)$$

$$= -\left(\frac{Q^2}{4\pi\varepsilon_0(4b^2)} - \frac{Q^2}{4\pi\varepsilon_0}\left(\frac{2b}{(a^2+4b^2)^{3/2}}\right)\right)$$

$$\cancel{= \frac{-Q^2}{4\pi\varepsilon_0}\left(\frac{1}{4b^2} - \frac{2b}{(a^2+4b^2)^{3/2}}\right)}$$

$$= \boxed{\frac{Q^2}{4\pi\varepsilon_0}\left(\frac{2b}{(a^2+4b^2)^{3/2}} - \frac{1}{4b^2}\right)}$$

For $-Q$.



By symmetry:

$$\cancel{F_x^- = \frac{-Q^2}{4\pi\varepsilon_0}\left(\frac{1}{a^2} - \frac{a}{(a^2+4b^2)^{3/2}}\right)}$$

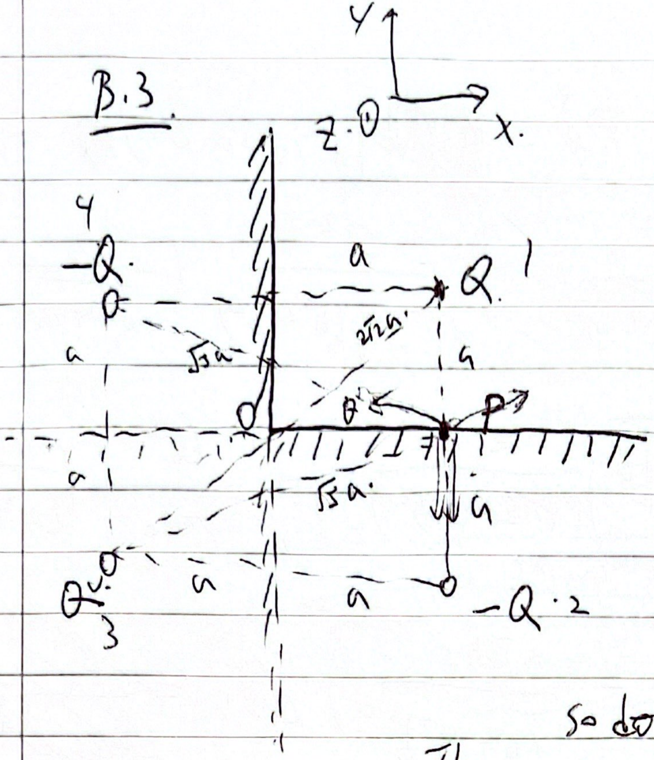
$$\cancel{F_y^- = \frac{-Q^2}{4\pi\varepsilon_0}\left(\frac{1}{4b^2} - \frac{2b}{(a^2+4b^2)^{3/2}}\right)}$$

$$\boxed{F_x^- = \frac{Q^2}{4\pi\varepsilon_0}\left(\frac{a}{(a^2+4b^2)^{3/2}} - \frac{1}{a^2}\right)}$$

$$\cancel{F_y^- = \frac{Q^2}{4\pi\varepsilon_0}\left(\frac{1}{4b^2} - \frac{2b}{(a^2+4b^2)^{3/2}}\right)}$$

$$\boxed{F_y^- = \frac{Q^2}{4\pi\varepsilon_0}\left(\frac{2b}{(a^2+4b^2)^{3/2}} - \frac{1}{4b^2}\right)}$$

B.3.

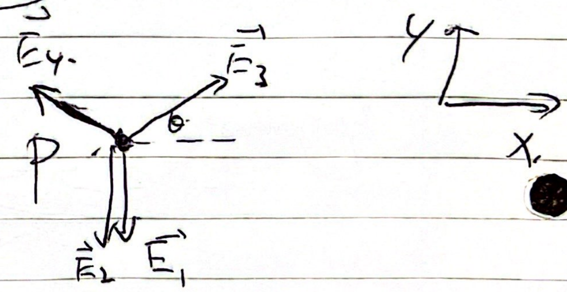


The induced charges on the two ~~plane~~ plates can be replaced by the three image charges shown.

∴ (a) The field of two Q's cancels each other along the join, so do the field of two −Q's. Thus the total Field E_0 along the join is 0.

(b) P is the required point.

By symetry $E_z = 0$



$$E_1 = E_2 = \frac{Q^2}{4\pi\varepsilon_0 a^2}.$$

$$E_3 = E_4 = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{5a^2}. \qquad \sin\theta = \frac{1}{\sqrt{5}}.$$

By symetry : $E_x = E_3\cos\theta - E_4\cos\theta = 0.$

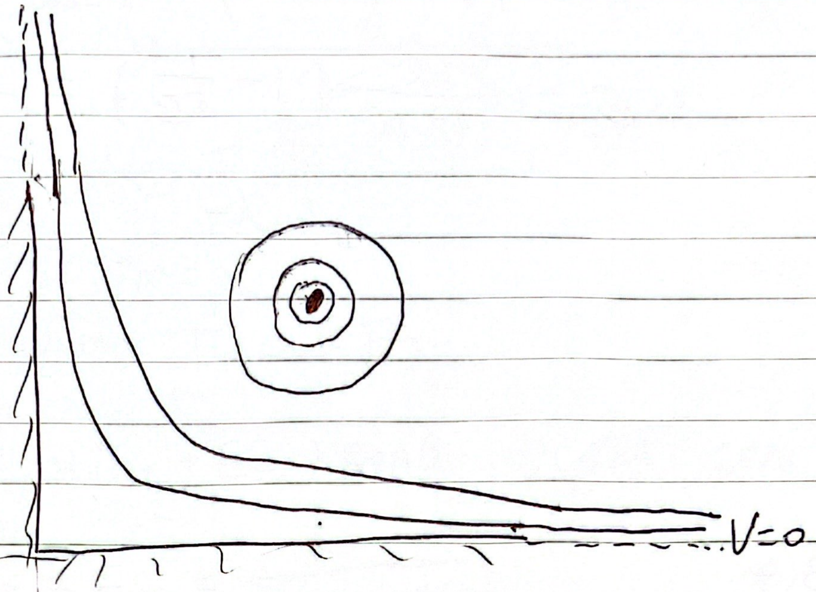
$$E_y = -(2E_1 - 2E_3\sin\theta)$$

$$= -\frac{2Q^2}{4\pi\varepsilon_0}\left(\frac{1}{a^2} - \frac{1}{5a^2}\left(\frac{1}{\sqrt{5}}\right)\right).$$

$$= -\frac{2Q^2}{4\pi\varepsilon_0}\left(\frac{1}{a^2} - \frac{1}{5\sqrt{5}a^2}\right)$$

$$\therefore \vec{E} = \left(0, -\frac{2Q^2}{4\pi\varepsilon_0 a^2}\left(1 - \frac{1}{5\sqrt{5}}\right), 0 \right)$$

(c)



$\oint E \cdot dA = \frac{Q_{encl}}{\varepsilon_0}$

$Q = \sigma A$

(d) $\sigma = \varepsilon_0 E_\perp$

$\therefore$ At point P, $E_\perp = E_y = \frac{2Q^2}{4\pi\varepsilon_0 a^2}\left(1 - \frac{1}{5\sqrt{5}}\right)$

$$\therefore \sigma = \frac{2Q^2}{4\pi a^2}\left(1 - \frac{1}{5\sqrt{5}}\right) = \boxed{\frac{Q^2}{2\pi a^2}\left(1 - \frac{1}{5\sqrt{5}}\right)}$$

B.4

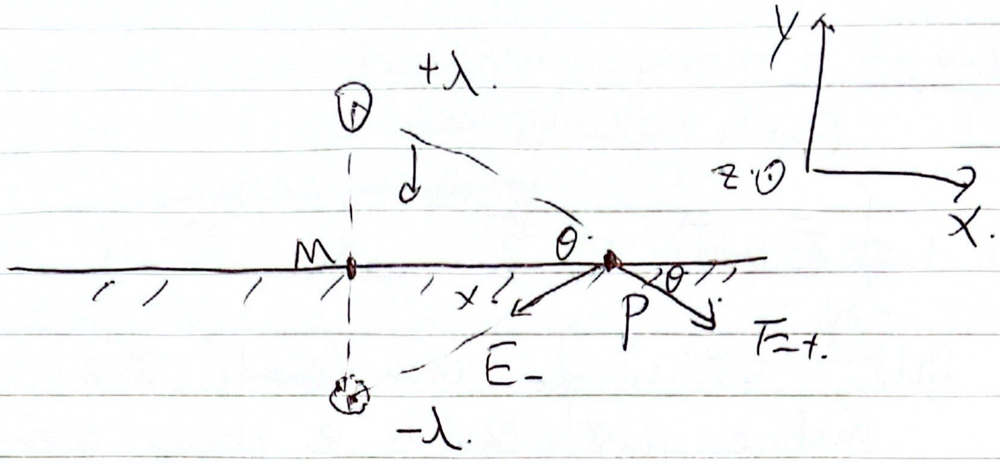
According to A.4

For a rod of length $2l$, the E-field at distance x · ~~per~~ above its mid-point is.

$$E = \frac{2l\lambda}{4\pi\varepsilon_0 d\sqrt{d^2+l^2}}$$

$$\lim_{l \to \infty} E = \lim_{l \to \infty} \frac{2l\lambda}{4\pi\varepsilon_0 d l\sqrt{1+\frac{d^2}{l^2}}} = \frac{2l\lambda}{2 \cdot 4\pi\varepsilon_0 d l} = \frac{\lambda}{2\pi\varepsilon_0 d}$$

$\therefore$ the E-field produced by a infinitely long wire is $E = \frac{\lambda}{2\pi\varepsilon_0 d}$, where x is the distance to wire.



let $PM = x$.

By symetry $E_x = 0 = E_z$.

$$E_y = (E_+ + E_-)\sin\theta \quad , \text{where } \sin\theta = \frac{d}{\sqrt{x^2+d^2}}.$$

$$E_+ = E_- = \frac{\lambda}{2\pi\varepsilon_0\sqrt{x^2+d^2}}$$

$$\therefore E_y = 2\left(\frac{\lambda}{2\pi\varepsilon_0\sqrt{x^2+d^2}}\right)\left(\frac{d}{\sqrt{x^2+d^2}}\right).$$

$$= \boxed{\frac{\lambda d}{\pi\varepsilon_0(x^2+d^2)}}$$

~~where x = PM.~~

pointing to $-\hat{e}_y$ direction

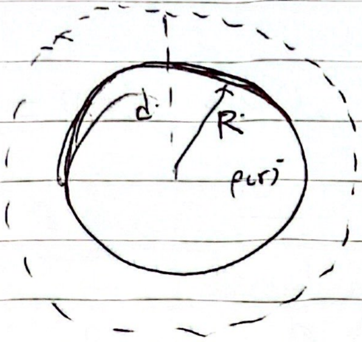
$$\therefore \vec{E} = \left(0, \frac{-\lambda d}{\pi\varepsilon_0(x^2+d^2)}, 0\right) \quad \text{where } x = \overline{PM}$$

6.0

Gauss's Law is

$$\oint_{\partial V} \vec{E}\cdot d\vec{s} = \frac{Q_V}{\varepsilon_0}$$

where $\vec{E}$ is the electric field, $d\vec{s}$ is the area element, ∂V is the surface and Q_V is the charge ~~on~~ enclosed by the surface. ε_0 is the permittivity.



For the spherically symmetric charge distribution of radius R.

$$Q_V = \int dQ = \int_0^r \rho(r)\, dV$$

$$= \int_0^R \rho(r)\,(4\pi r^2)\,dr.$$

$$\oint \vec{E}\cdot d\vec{s} = E(4\pi d^2)$$

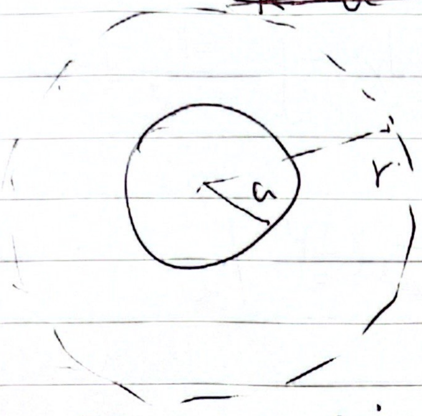
$$\cancel{\therefore E = \frac{\int_0^R \rho(r)}{4\pi\varepsilon_0 r^2}}$$

$$\therefore E = \frac{4\pi\int_0^R r^2\rho(r)\,dr}{4\pi\varepsilon_0 d^2}$$

$$\therefore \boxed{E = \frac{\int_0^R r^2\rho(r)\,dr}{\varepsilon_0 d^2}}$$

(a) Electric field always point to $\hat{r}$ due to spherically symetric distribution of charges

when $a \le r$.



Gaussian Surface.

Charge enclosed $Q_v = q$.

$$\therefore \oint_{\partial v} \vec{E} \cdot d\vec{s} = \frac{Q_v}{\varepsilon_0}$$

$$\Rightarrow E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$\therefore \boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}}$$

$$\vec{E} = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r} = -\int_{\infty}^{r} \vec{E} \cdot \hat{r}\, dr$$

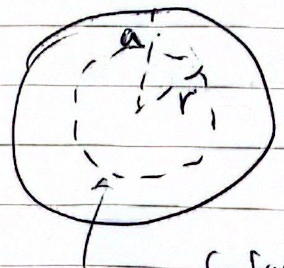
$$= -\int_{\infty}^{r} \frac{q}{4\pi\varepsilon_0 r'^2}\, dr' = \frac{-q}{4\pi\varepsilon_0}\left[-\frac{1}{r'}\right]\Bigg|_{\infty}^{r}$$

$$= \boxed{\frac{q}{4\pi\varepsilon_0 r}}$$

when $0 \le r \le a$.

charge density $\rho = \frac{q}{\frac{4}{3}\pi a^3} = \frac{3q}{4\pi a^3}$ Charge enclosed $Q_v = \rho(\frac{4}{3}\pi r^3) = \frac{3q}{4\pi a^3}(\frac{4}{3}\pi r^3)$

$$= \frac{qr^3}{a^3}$$



Gaussian Surface.

$$\therefore \oint_{\partial v} \vec{E} \cdot d\vec{s} = \frac{Q_v}{\varepsilon_0} \Rightarrow E(4\pi r^2) = \frac{qr^3}{a^3\varepsilon_0}$$

$$\therefore \boxed{\vec{E} = \frac{qr}{4\pi\varepsilon_0 a^3}\hat{r}}$$

$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r} = -\int_{\infty}^{r} \vec{E} \cdot \hat{r}\, dr$$

$$= -\int_{\infty}^{r} E\, dr = -\int_{\infty}^{a} \frac{q}{4\pi\varepsilon_0 r'^2}\, dr' - \int_{a}^{r} \frac{q r'}{4\pi\varepsilon_0 a^3}\, dr'$$

$$= \frac{-q}{4\pi\varepsilon_0}\left(-\frac{1}{r'}\right)\Big|_{\infty}^{a} - \frac{q}{4\pi\varepsilon_0 a^3}\left(\frac{1}{2}r'^2\right)\Big|_{a}^{r}$$

$$= \frac{q}{4\pi\varepsilon_0 a^3}(a^2) - \frac{q}{4\pi\varepsilon_0 a^3}\left(\frac{1}{2}r^2 - \frac{1}{2}a^2\right)$$

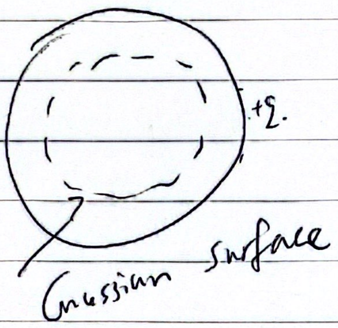
$$= \frac{q}{4\pi\varepsilon_0 a^3}\left(\frac{3}{2}a^2 - \frac{1}{2}r^2\right)$$

(b) For surface of sphere.

when $a \le r$, same argument in (a) applies.

$$\therefore\ E = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}, \quad V = \frac{q}{4\pi\varepsilon_0 r}$$

When $0 \le r \le a$.



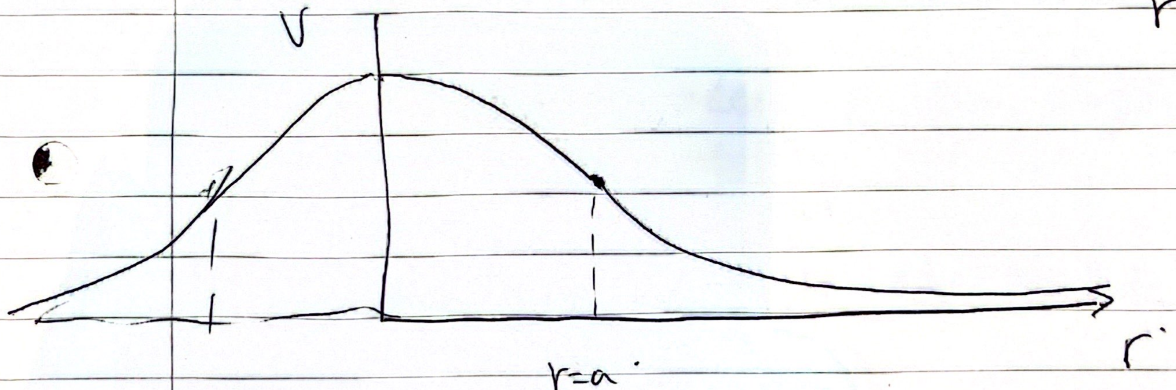
the charged enclosed is 0.

$$\therefore\ \boxed{\vec{E} = 0}$$

$$V = -\int_{\infty}^{a} \frac{q}{4\pi\varepsilon_0 r^2}\, dr - \int_{a}^{r} 0\, dr.$$

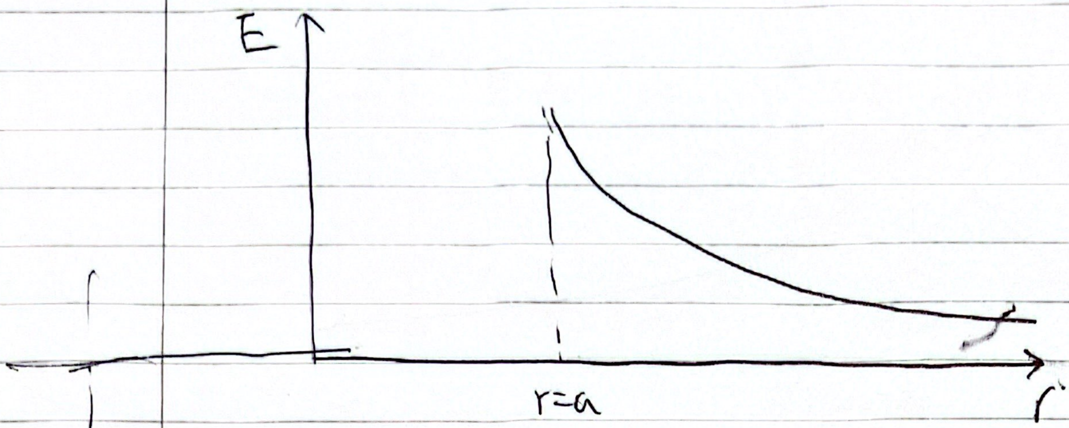
$$= \boxed{\frac{q}{4\pi\varepsilon_0 a}}$$

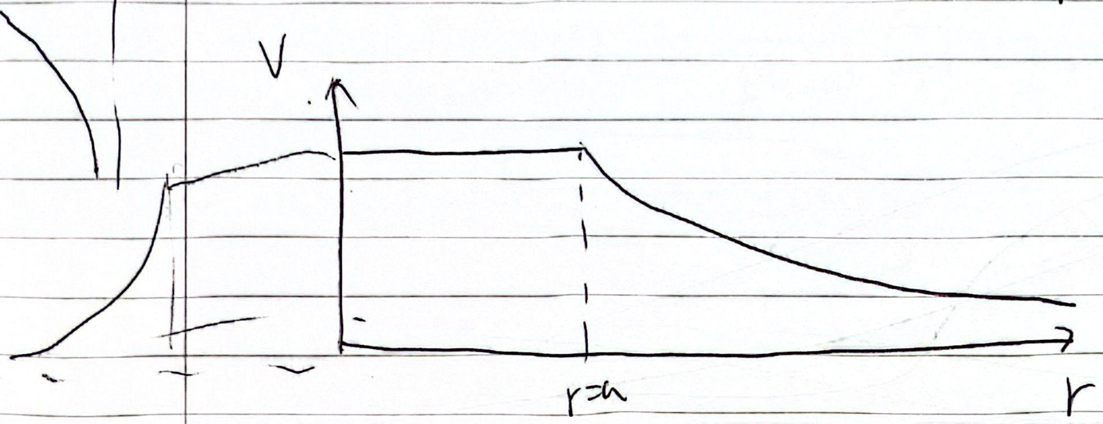
(c) For solid sphere.

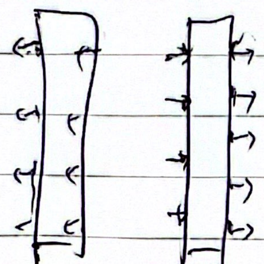


~~for~~

For surface of sphere.



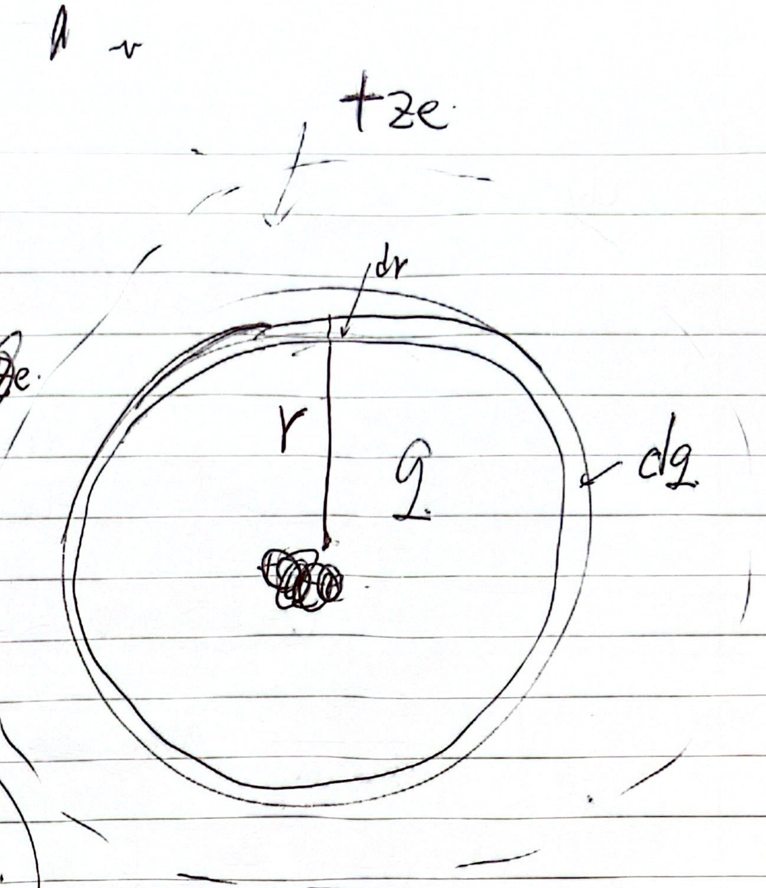




un polarized.

C.2

(a)



We peel off the charge layer by layer (and do work to each layer to move them to infinity, from a to 0)

$$\therefore W = \int \frac{q\,dq}{4\pi\varepsilon_0 r}$$

$\because$ sphere has uniform distribution of charges

$\therefore$ $q(r) = \frac{r^3}{a^3} Ze$, volume charge density

$$\rho = \frac{Ze}{\frac{4}{3}\pi a^3} = \frac{3Ze}{4\pi a^3}$$

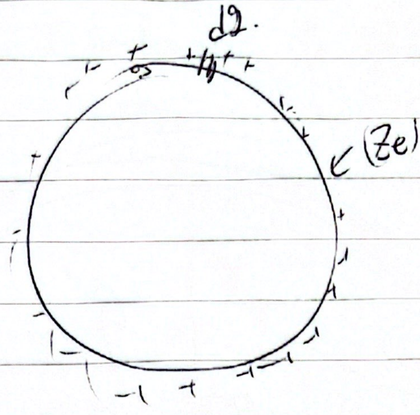
charge of one layer $dq = \rho\, dV = 4\pi r^2 \rho\, dr = \frac{3Ze r^2}{a^3} dr$.

$$\therefore W = \int_0^a \frac{[r^3(Ze)/a^3].[3(Ze)r^2/a^3]\,dr}{4\pi\varepsilon_0 r}$$

$$= \frac{3(Ze)^2}{4\pi\varepsilon_0 a^6}\int_0^a r^4\,dr = \frac{3(Ze)^3}{4\pi\varepsilon_0 a^6}\left[\frac{r^5}{5}\right]_0^a$$

$$= \frac{3(Ze)^2}{4\pi\varepsilon_0 a^6}\left(\frac{a^5}{5}\right) = \boxed{\frac{3(Ze)^2}{20\pi\varepsilon_0 a}}$$

(b)



The total potential energy of the distribution is given by

$$U = \frac{1}{2}\sum q_i V_i = \frac{1}{2}\oint_{\partial V} V\,dq.$$

(∂V is the surface of sphere)

$V = \frac{Ze}{4\pi\varepsilon_0 a}$ ~~along~~ for any point on the sphere

$\therefore$ $U = \frac{Ze}{8\pi\varepsilon_0 a}\oint_{\partial V} dq = \frac{Ze}{8\pi\varepsilon_0 a}(Ze) = \boxed{\frac{Z e^2}{8\pi\varepsilon_0 a}}$

~~(c)~~

~~(a)~~

~~V~~

~~the curve starts from the point $V = \frac{q}{4\pi\varepsilon_0 a}$ and decays like an exponential decay except it crosses zero at some point of r and then approachs 0 as r goes to ∞ from negative values of V.~~

~~r~~

C.3

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{e^{(-2r/a)}-1}{r} + \frac{e^{-2r/a}}{a}\right)$$

L'hopital rule → $\lim_{r\to 0} V = \frac{q}{4\pi\varepsilon_0}\left(\frac{-\frac{2}{a}e^{-2r/a}}{1} + \frac{e^{-2r/a}}{a}\right)$

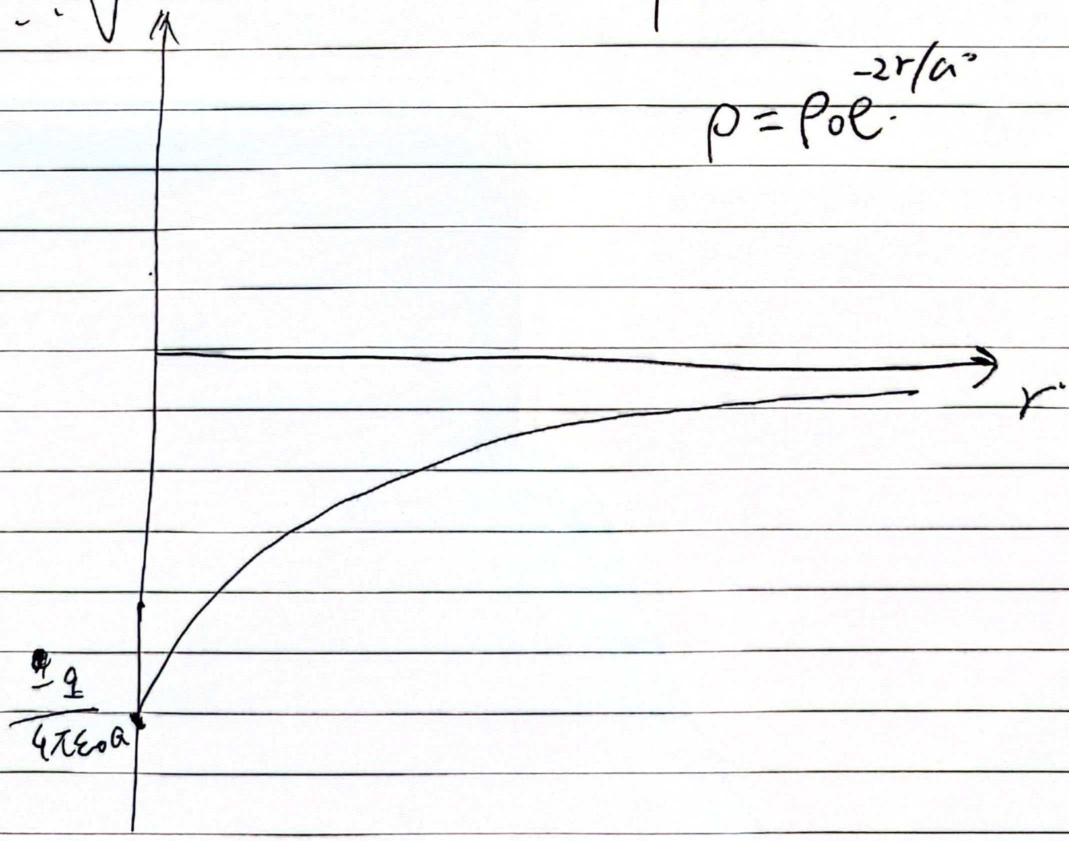
$$= \frac{q}{4\pi\varepsilon_0}\left(-\frac{2}{a}+\frac{1}{a}\right) = \frac{-q}{4\pi\varepsilon_0 a}$$

$$\lim_{r\to\infty} V = (0+0) = 0$$

$$\varphi = e^{-r/a_0}$$

$$\rho = \rho_0 e^{-2r/a_0}$$

$\therefore$



(b)

$$\because V = V(\vec{r}) \quad \therefore \quad \vec{E} = -\frac{\partial V}{\partial r}\hat{r}$$

$$\therefore \vec{E} = -\frac{\partial V}{\partial r} = \frac{-q\hat{r}}{4\pi\varepsilon_0}\left[\frac{1}{r^2} + \left(\frac{r(-\frac{2}{a}e^{-2r/a}) - (e^{-2r/a})(1)}{r^2}\right) - \frac{2}{a^2}e^{-2r/a}\right]$$

$$= \frac{-q\hat{r}}{4\pi\varepsilon_0}\left[\frac{1}{r^2} - \frac{2}{ra}e^{-2r/a} - \frac{1}{r^2}e^{-2r/a} - \frac{2}{a^2}e^{-2r/a}\right]$$

$$= \frac{-q\hat{r}}{4\pi\varepsilon_0 r^2}\left[1 - e^{-2r/a}\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)\right]$$

$\therefore$ ~~how~~ magnitude of $\vec{E}$ is

$$E = \frac{q}{4\pi\varepsilon_0 r^2}\left[1 - e^{-2r/a}\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)\right]$$

For $r \ll a$, we expand $e^{-2r/a}$ to be.

$$e^{-2r/a} = 1 - \frac{2r}{a} + \left(\frac{2r}{a}\right)^2/2 + \left(-\frac{2r}{a}\right)^3/6 + \cdots$$

$$= 1 - 2\left(\frac{r}{a}\right) + 2\left(\frac{r}{a}\right)^2 - \frac{4}{3}\left(\frac{r}{a}\right)^3 + \cdots$$

we know $\left(1 - 2\left(\frac{r}{a}\right) + 2\left(\frac{r}{a}\right)^2 - \frac{4}{3}\left(\frac{r}{a}\right)^3\right)\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)$

$$= 1 \cancel{-2\left(\frac{r}{a}\right)} \cancel{+2\left(\frac{r}{a}\right)^2} - \frac{4}{3}\left(\frac{r}{a}\right)^3 + \cancel{\frac{2r}{a}} \cancel{-4\left(\frac{r}{a}\right)^2} + 4\left(\frac{r}{a}\right)^3 - \frac{8}{3}\left(\frac{r}{a}\right)^4$$
$$+ \cancel{2\left(\frac{r}{a}\right)^2} - 4\left(\frac{r}{a}\right)^3 + 4\left(\frac{r}{a}\right)^4 - \frac{8}{3}\left(\frac{r}{a}\right)^5$$

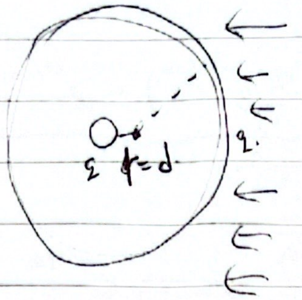
all $\left(\frac{r}{a}\right)$, $\left(\frac{r}{a}\right)^2$ cancells and we ~~also~~ ignore all $\left(\frac{r}{a}\right)^4$, $\left(\frac{r}{a}\right)^5$ terms for $r \ll a$.

$\therefore$ the original expression becomes $1 - \frac{4}{3}\left(\frac{r}{a}\right)^3$

$$\therefore E = \frac{q}{4\pi\varepsilon_0 r^2}\left[1 - \left(1 - \frac{4}{3}\left(\frac{r}{a}\right)^3\right)\right] = \frac{q}{4\pi\varepsilon_0 r^2}\left(\frac{4}{3}\frac{r^3}{a^3}\right)$$

$$= \boxed{\frac{qr}{3\pi\varepsilon_0 a^3}}$$

(c) when external field E_{ext} is applied.



$q E_{ext}$

$\leftarrow \ominus \rightarrow$

qE

Balancing forces on the nucleus gives $qE_{ext} = qE$ $\therefore E_{ext} = E$

$$\therefore \quad E_{ext} = \frac{qd}{3\pi\varepsilon_0 a^3}$$ where $r = d$ is the separation ~~of~~ between the nucleus and the centre of the cloud at equilibrium. $p = qd$ is the dipole moment

$$\therefore \quad \boxed{p = 3\pi\varepsilon_0 a^3 E_{ext}} \quad \text{and} \quad \boxed{\frac{p}{E_{ext}} = 3\pi\varepsilon_0 a^3}$$

(d)

$a = 0.5 \times 10^{-10}$ m $\quad E_{ext} - 10^6$ V m^{-1} $\quad q = 1.6 \times 10^{-19}$ C.

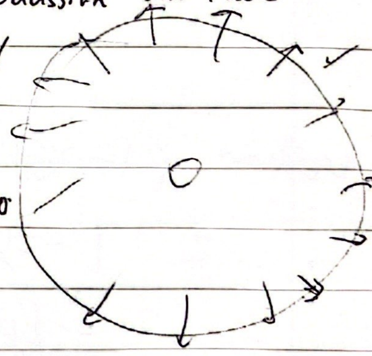
$\varepsilon_0 = 8.854 \times 10^{-12}$ F. m^{-1}.

$$d = \frac{3\pi\varepsilon_0 a^3 E_{ext}}{q} = 6.5 \times 10^{-17} m \sim \boxed{10^{-17} m}$$

(e) Gauss's Law

$$\oint_{\partial V} \vec{E} \cdot ds = \frac{Q_V}{\varepsilon_0}$$

Charge distribution extends to infinity so sketch Gaussian surface near infinity



$$E = \frac{-q}{4\pi\varepsilon_0 r^2} \left(1 - e^{-2r/a}\left(1 + \frac{2r}{a} + \frac{2r^2}{a^2}\right)\right)$$

when $r \to \infty$ $\quad \lim_{r\to\infty} e^{-2r/a} r^2 = \lim_{r\to\infty} e^{-2r/a} r = \lim_{r\to\infty} e^{-2r/a} = 0$

$$\therefore \lim_{r\to\infty} E = \frac{-q}{4\pi\varepsilon_0 r^2}.$$

$$\therefore \quad \frac{-q}{4\pi\varepsilon_0 r^2} \cdot (4\pi r^2) = \frac{Q_V}{\varepsilon_0} \quad \therefore \quad Q_V = \boxed{q}.$$

$\therefore$ total charge is $\boxed{-q}$