

Classical and Quantum Solitons.

$$z \rightarrow 120^\circ. R_\psi(z) \rightarrow e^{\frac{2i\pi}{3}} R_\psi(z).$$

Soliton Lect

Classical and Quantum Solitons.0. Introductory Remarks.

Solitons are stable, spatially localised, smooth solutions of classical field equations in some QFT.

In relativistic theory, they have energy E (= mass M if at rest)

and momentum P , satisfying $E^2 - \underline{P} \cdot \underline{P} = m^2$
so they are particles.

Solve eqns. exactly, quantized approximately

Requirement: 1. Non linear equations
or Non linear field structure.

and often 2. Multiple vacuum. w/
~~nontrivial~~ non-trivial topology.
from symmetry breaking.

3. Gauge Symmetry.

Interesting solitons have topological
stability and ~~to~~ interact w/ particles.

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Contrast perturbative QFT

1. unique vacuum
2. particles are quantised waves
Have E and p related to m
only after quantisation
3. Quantisation exact,
nonlinearities treated approximation

This course: 1D kinks

2D vortices

Topology - degree of a map.

2D σ -model solitons - rational maps.

3D steady state solitons.

- applications to nucleons / nuclei
(protons / neutrons)

1 kinks

1+1 dimension, non linearity due to potential U with multiple vacua.

Scalar field $\phi(x,t)$, ϕ

Lagrangian $L = \int_{-\infty}^{\infty} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \right) dx$

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Action: $S[\phi] = \int L dt = \int \mathcal{L} dx dt$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi)$$

Note: Split $L = T - V$

$$T = \int \frac{1}{2} \dot{\phi}^2 dx$$

$$V = \int \phi \left(\frac{1}{2} \phi'^2 + U(\phi) \right) dx$$

$$\dot{\phi} = \frac{\partial \phi}{\partial t} \quad \phi' = \frac{\partial \phi}{\partial x}$$

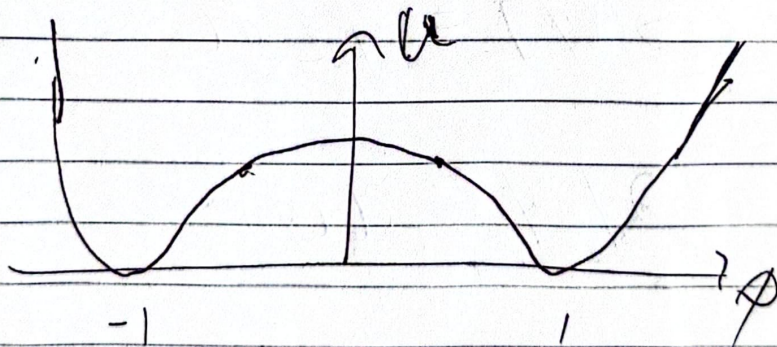
$S[\phi]$ stationary under $\phi \rightarrow \phi + \delta\phi$
gives field eq.

$$\partial_\mu \partial^\mu \phi + \frac{dU}{d\phi} = 0 \quad \text{Non-linear KG eq.}$$

For a static solution. $(\frac{\partial}{\partial t} = 0)$.

$$\frac{d^2\phi}{dx^2} = -\frac{dU}{d\phi}.$$

ϕ^4 theory choose $U = \frac{1}{2}(1 - \phi^2)^2$



$$[\text{or } \lambda(m^2 - \phi^2)^2, \lambda\phi^2(m^2 - \phi^2)^2, \\ 1 - \cos\phi]$$

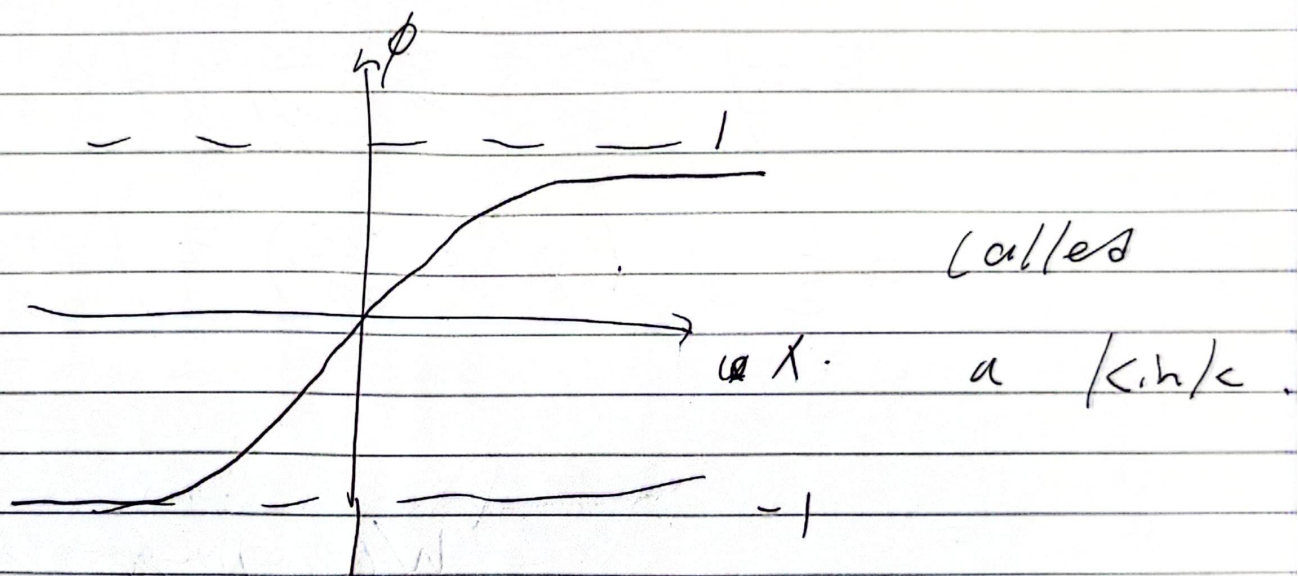
these are the vacuum field configurations

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$$\begin{aligned} \int \phi(x) &= 1 \\ \phi(x) &= -1 \end{aligned} \quad \text{with zero energy.}$$

Field eq. $\frac{d^2\phi}{dx^2} = -2(1-\phi^2)\phi$

has further solution connecting these vacua.



(anti-kink goes the other way)

Find Kink by considering energy.
(rather than solving directly).

Conserved energy - (From Noether's theorem)

$$E = \int \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi'^2 + U(\phi) \right) dx.$$

For static field $E = \int \left(\frac{1}{2} \phi'^2 + U(\phi) \right) dx.$
(in general here)

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Now assume we can express -

$$U(\phi) = \frac{1}{2} \left(\frac{dw}{d\phi} \right)^2 \quad (\text{can do this for } w \geq 0)$$

Bogomolny rearrangement -

$$E = \frac{1}{2} \int (\phi'^2 + \left(\frac{dw}{d\phi} \right)^2) dx.$$

$$= \frac{1}{2} \int \left(\phi' \mp \frac{dw}{d\phi} \right)^2 \pm \int \underbrace{\frac{dw}{d\phi} \frac{d\phi}{dx}}_{\frac{dw}{dx}} dx.$$

$$= \quad \quad \quad \pm \left(W(\phi(\infty)) - W(\phi(-\infty)) \right)$$

Depend on boundary conditions
"topological"

(can't jump from one to the other)

E is minimized when $\phi' = \pm \frac{dw}{d\phi}$,

and E has the value $E = \pm (W(\infty) - W(-\infty))$

Bogomolny eq. $E = \pm (W(\infty) - W(-\infty))$
energy bound.

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Solitons Lec 2
Bogomolny rearrangement.

$$\phi' = \frac{d\phi}{dx}$$

$$E = \frac{1}{2} \int (\phi'^2 + (\frac{dw}{d\phi})^2) dx$$

→ (static kink)

$$U = \frac{1}{2} (\frac{dw}{d\phi})^2$$

$$= \frac{1}{2} \int (\phi' \mp \frac{dw}{d\phi})^2 dx \pm (W(\phi(\infty)) - W(\phi(-\infty)))$$

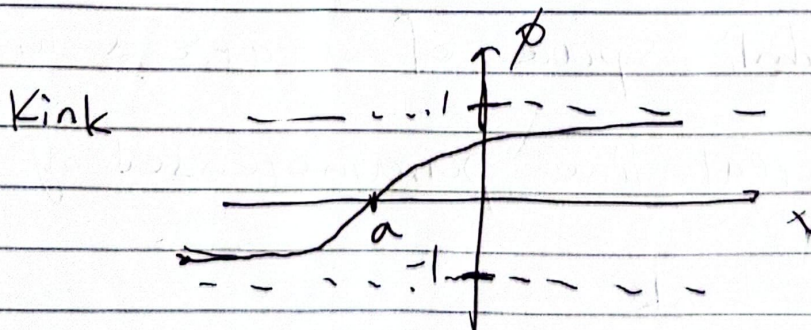
E is minimized when $\phi' = \pm \frac{dw}{d\phi}$ (Bog. eq.)
and has value

$$E = \pm (W(\infty) - W(-\infty)) \quad (\text{topological value})$$

ϕ^4 - Kink

$$U = \frac{1}{2} (1 - \phi^2)^2$$

$$\frac{dw}{d\phi} = 1 - \phi^2, \quad W = \phi - \frac{1}{3}\phi^3$$



when $\phi = 1, W = \frac{2}{3}$

$\phi = -1, W = -\frac{2}{3}$

for +ve E

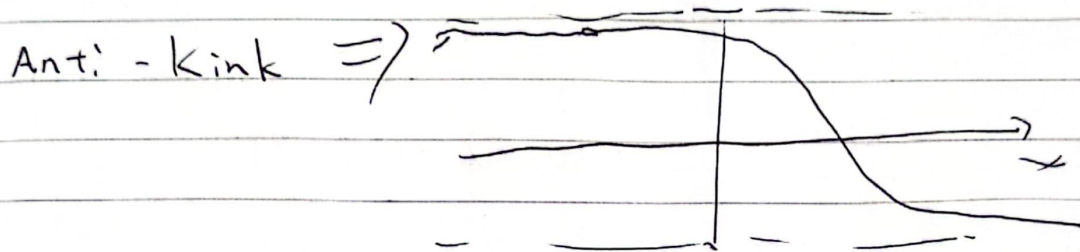
Energy of Kink $E = + (\frac{2}{3} - (-\frac{2}{3}))$
 $= \frac{4}{3} = M$ (static kink)

(2)

Eq. for kink $\frac{d\phi}{dx} = 1 - \phi^2$

Kink solution: $\phi(x) = \tanh(x-a)$

a is an arbitrary constant - location of the kink.



Energy $E = +\frac{4}{3}$ $E = -(-\frac{2}{3} - (-\frac{2}{3}))$

satisfies $\frac{d\phi}{dx} = -(1 - \phi^2)$

" a " is the modulus of the kink solution

The moduli space of static solution is

$\mathbb{R}$. (real line parameterised by a)

Dynamic Kink

There is an exact kink soln (orentz boosted)

$$\phi(x,t) = \tanh(\gamma(x-vt))$$

$$\gamma = (1-v^2)^{-1/2}$$

(3)

very useful is an adiabatic point of view. (slowly moving)

Consider $\phi(x,t) = \tanh(x - a(t))$

$$\text{so } \dot{\phi} = - \frac{da}{dt} \phi'$$

Find dynamics of $a(t)$ from Lagrangian of field theory

$$T = \int \frac{1}{2} \dot{\phi}^2 dx = \frac{1}{2} M \left(\frac{da}{dt} \right)^2$$

$$\text{where } M = \frac{4}{3}$$

(kinetic energy)

$$\text{and } V = \text{const} = \frac{4}{3}$$

Reduced Lagrangian

$$E = T + V$$

$$\min(T) = 0$$

$$\therefore \min(E) = \frac{4}{3}$$

$$L = \frac{1}{2} M \dot{a}^2 - \text{const}$$

This is for 1-dim motion in a valley

in ~~so-called~~ ∞ -dim field config.

effective eq. of motion in moduli space

$$M \ddot{a} = 0 \quad \therefore a(t) = Vt + \text{const}$$

$\checkmark$ const, as before.

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but missing Lorentz contraction

Motion in moduli space not exact, but ok in non-relativistic sense.

Motion in moduli space will allow ^{analysis} ~~over~~ of motion of several solutions (vortices etc,

in higher dimensions, where simple symmetry (Lorentz symmetry) not available.

Momentum In 1+1 dim, single scalar field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - u(\phi)$$

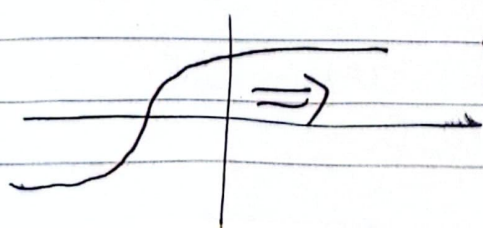
E-M tensor is $T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial^\mu \phi - \delta^{\mu\nu} \mathcal{L}$

Conserved energy.

$$E = \int_{-\infty}^{\infty} T^0_0 dx = \int_{-\infty}^{\infty} \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi'^2 + u(\phi) \right) dx$$

Conserved momentum

$$P = - \int_{-\infty}^{\infty} T^0_1 dx = - \int_{-\infty}^{\infty} \dot{\phi} \phi' dx$$



$\dot{\phi} < 0 \Rightarrow$ right moving
Kink the non.

(5)

In moduli space analysis $\phi = \tanh(x - a(t))$
 $\Rightarrow \dot{\phi} = -\phi' \frac{da}{dt}$

$$p = \frac{da}{dt} \int_{-\infty}^{\infty} \phi'^2 dx = M \frac{da}{dt}$$

Quantisation is very simple using moduli space approximation.

Free particle with position a .

Classical Hamiltonian. $P = M\dot{a}$

$$H = P\dot{a} - L = \frac{1}{2m} P^2$$

Quantisation $P \rightarrow -i\hbar \frac{\partial}{\partial a}$

$$H \rightarrow -\hbar^2 \frac{\partial^2}{\partial a^2}$$

Wavefunction, a function of ~~a~~ a and t .

Ordinary Q.M. for a particle with position a

Stationary states $\psi(a) = e^{i k a}$.

Q & ψ values $P = \hbar k$, $H = \frac{\hbar^2 k^2}{2m} = \frac{P^2}{2m}$.

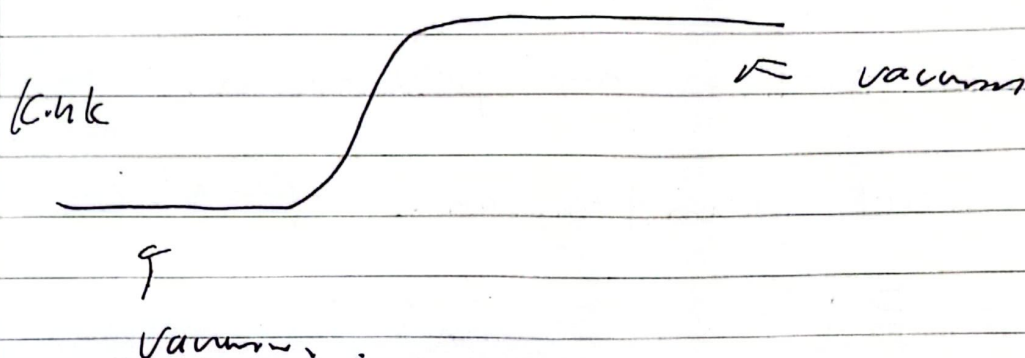
Remark "a" is called modulus of

collective coordinate of Kink.

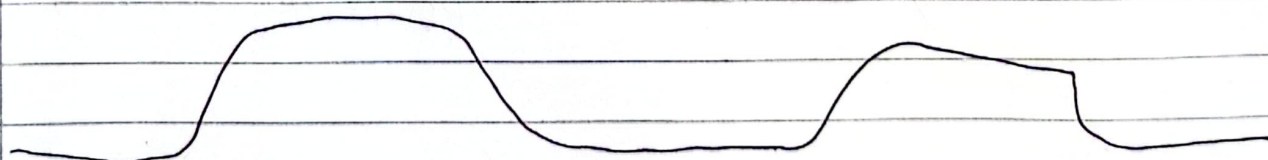
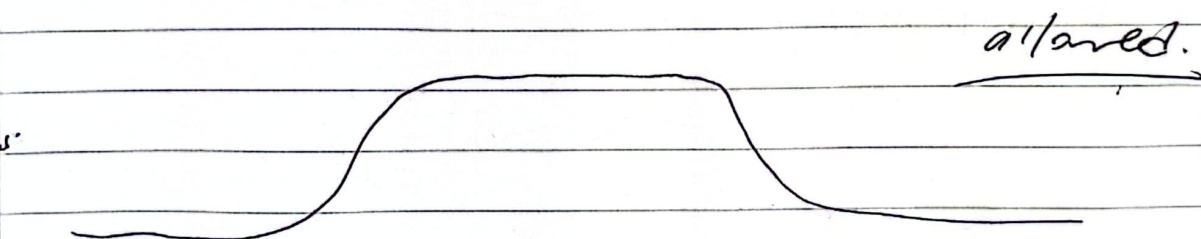
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Classical and Quantum dynamics of a system is called collective coord. dynamics.

Quantisation of complete field theory (eg Rajaraman, Weinberg) gives kinks or ~~then~~ Mesons. (Quantized waves around ~~vac~~ vacuum)



full
Quantum
theories.



Field oscillations (quantized) around the.
Kink soliton $\Rightarrow$ mesons interacting with the soliton.

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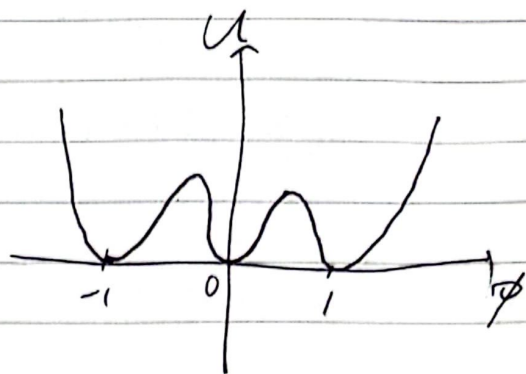
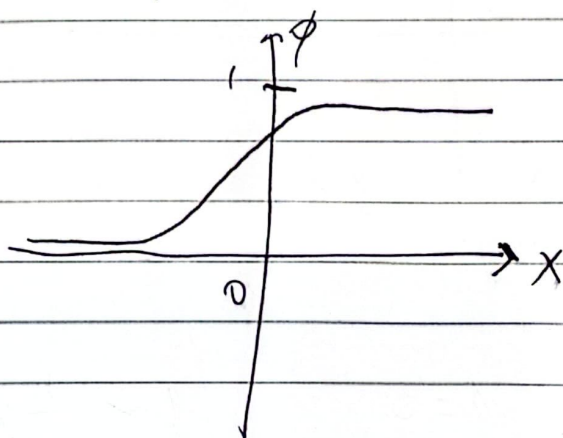
There is a quantum correction (small)
to the mass M of the kink.

(1)

Solitons Lec 3

 ϕ^6 kink

$$U(\phi) = \frac{1}{2} (1 - \phi^2)^2 \phi^2$$

Kink from $\phi = 0$ to $\phi = 1$ 

$$U = \frac{1}{2} \left(\frac{dw}{d\phi} \right)^2 \quad \text{where} \quad \frac{dw}{d\phi} = (1 - \phi^2)\phi$$

$$\text{so } w(\phi) = \frac{1}{2} \phi^2 - \frac{1}{4} \phi^4 + \text{const.}$$

$$\text{Bog. eq. } \frac{d\phi}{dx} = (1 - \phi^2)\phi \quad \left(\text{need "4"} \right)$$

der kink flow eq.

$$\Rightarrow \left(\frac{1}{\phi} - \frac{\phi}{1 - \phi^2} \right) d\phi = dx.$$

$$\log \phi - \frac{1}{2} \log(1 - \phi^2) = x - a.$$

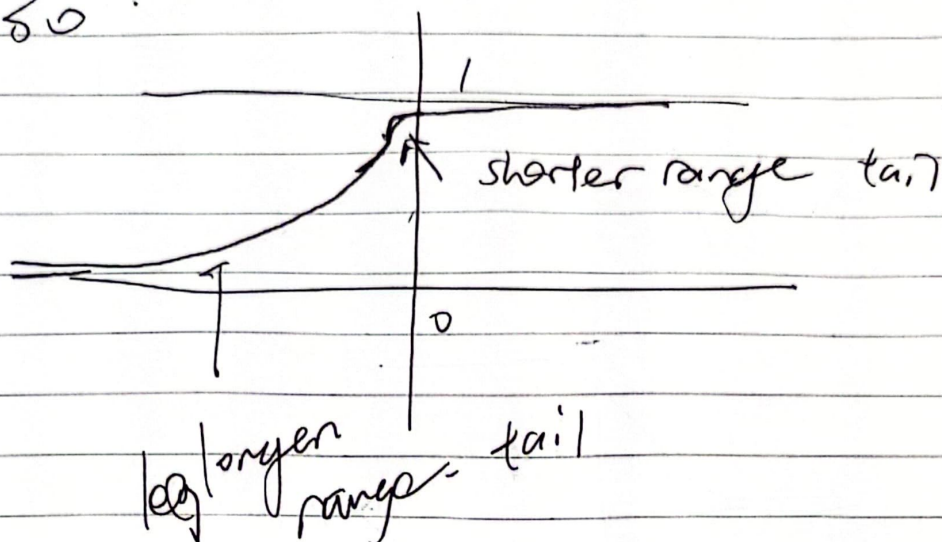
$$\frac{\phi^2}{1 - \phi^2} = e^{2(x - a)}$$

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$$\Rightarrow \phi(x) = (1 + e^{-2(x-a)})^{-\frac{1}{2}}$$

(ϕ can't always be solved exactly as here)

So.



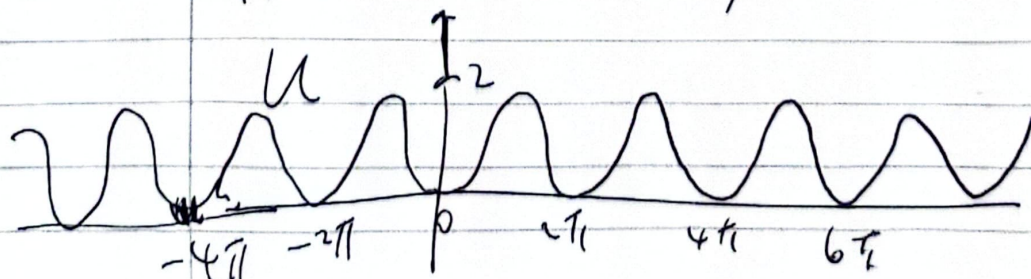
$$\text{Near } \begin{cases} \phi=0 & U \sim \frac{1}{2}\phi^2 \\ \phi=1 & U \sim 2(1-\phi)^2 \end{cases}$$

~~Ex~~ Energy (Mass) of Kink

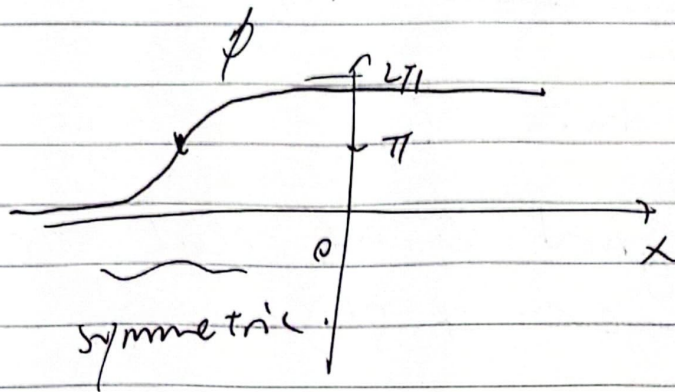
$$= W(1) - W(0) = \frac{1}{4}$$

- Sine-Gordon Solitons

Here $U = 1 - \cos\phi$



so $\frac{dw}{d\phi} = 2\sin(\frac{1}{2}\phi)$



1-kink Soliton

Static solutions only exist joining neighbouring vacua

$$\phi = 2n\pi \text{ to } 2(n \pm 1)\pi.$$

Best to identify $\phi \bmod 2\pi$, so $\phi: \mathbb{R} \rightarrow S^1$ at each instant.

Now a single vacuum $\phi = 0 \bmod 2\pi$ but a field non-linear.

The vacuum $\rightarrow$ boundary condition:

$$\phi = 0 \bmod 2\pi \text{ as}$$

$$x \rightarrow \pm \infty$$

Topologically, ϕ is a map $S^1_{\text{compactified 1 dim space}} \rightarrow S^1_{\text{target}}$.

(4)

(S^1 is $\mathbb{R}$ with ends identified.)

ϕ has a winding number Q .

How to find this?

1. (Naive) Regard ϕ as real variable $\in \mathbb{R}$
(universal cover of S^1)

Suppose $\left\{ \begin{array}{l} \phi(-\infty) = 2\eta_L \pi \\ \phi(\infty) = 2\eta_R \pi \end{array} \right.$

$$Q = \eta_R - \eta_L$$

Eg in figure $Q = 1$.

Q is the net soliton number
(# of kinks - # of anti-kinks)

Q is also called the topological charge

2. Noether (Topological current).

$$j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi$$

ϵ is antisymmetric tensor ($\epsilon^{01} = 1$ etc)

$$= \frac{1}{2\pi} (\partial_x \phi, -\partial_t \phi)$$

(5)

This is conserved

$$\partial_\mu j^\mu = \frac{1}{2\pi} \underbrace{\epsilon^{\mu\nu} \partial_\mu \partial_\nu \phi}_{=0} = 0$$

(by sym of mixed partials)

Charge' $Q = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\partial_x \phi}_{j^0} dx = \frac{1}{2\pi} (\phi(\infty) - \phi(-\infty))$

$= n_R - n_L$

(a total derivative
use is topological)

Q is an integer independent of time.

3. (~~Def Diff Form~~). Q is degree of the map.

$$\phi: S_\infty \rightarrow S'$$

Target S' has a normalised volume form
(length in 1-d) $\frac{1}{2\pi} d\phi$ (1-form).

So satisfies $\int_0^{2\pi} \frac{1}{2\pi} d\phi = 1$

If ϕ is a function of x i.e. $\phi = \phi(x)$.
then, pull-back of $\frac{1}{2\pi} d\phi$ is the

(6)

$$\frac{1}{2\pi} \frac{d\phi}{dx} dx \quad \text{on } \Sigma_{\infty}^1 \quad (\text{with coord } x).$$

Define the degree of map.

$$Q = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{d\phi}{dx} dx = \text{same as def in 2.}$$

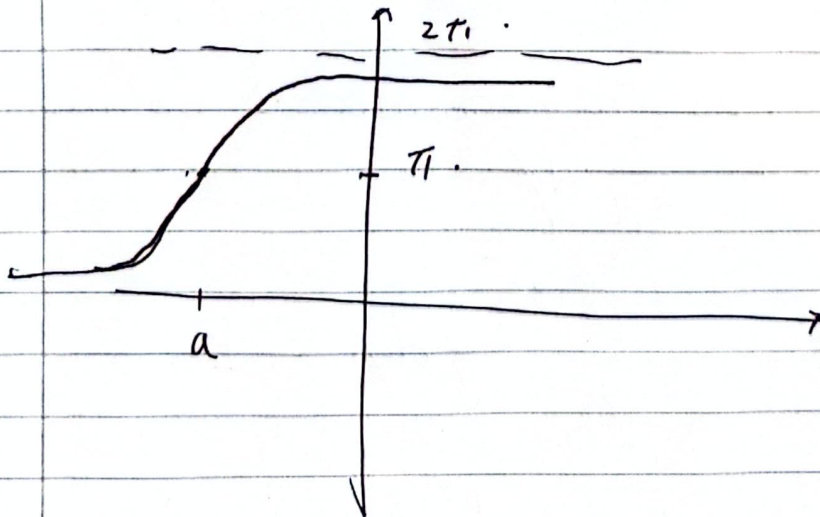
$$= n_R - n_L.$$

Generally the topological degree of a map is the integral of pull-back of a normalised volume form on the target of case (a) (b) (c) (d) (e) (f) (g) (h) (i) (j) (k) (l) (m) (n) (o) (p) (q) (r) (s) (t) (u) (v) (w) (x) (y) (z) (aa) (ab) (ac) (ad) (ae) (af) (ag) (ah) (ai) (aj) (ak) (al) (am) (an) (ao) (ap) (aq) (ar) (as) (at) (au) (av) (aw) (ax) (ay) (az) (ba) (bb) (bc) (bd) (be) (bf) (bg) (bh) (bi) (bj) (bk) (bl) (bm) (bn) (bo) (bp) (bq) (br) (bs) (bt) (bu) (bv) (bw) (bx) (by) (bz) (ca) (cb) (cc) (cd) (ce) (cf) (cg) (ch) (ci) (cj) (ck) (cl) (cm) (cn) (co) (cp) (cq) (cr) (cs) (ct) (cu) (cv) (cw) (cx) (cy) (cz) (da) (db) (dc) (dd) (de) (df) (dg) (dh) (di) (dj) (dk) (dl) (dm) (dn) (do) (dp) (dq) (dr) (ds) (dt) (du) (dv) (dw) (dx) (dy) (dz) (ea) (eb) (ec) (ed) (ee) (ef) (eg) (eh) (ei) (ej) (ek) (el) (em) (en) (eo) (ep) (eq) (er) (es) (et) (eu) (ev) (ew) (ex) (ey) (ez) (fa) (fb) (fc) (fd) (fe) (ff) (fg) (fh) (fi) (fj) (fk) (fl) (fm) (fn) (fo) (fp) (fq) (fr) (fs) (ft) (fu) (fv) (fw) (fx) (fy) 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So Start: Sine-Gordon Kink.

We need to solve $\frac{d\phi}{dx} = 2 \sin \frac{1}{2} \phi$ (Bogo eq.)

Solution is $\phi(x) = 4 \tan^{-1} e^{x-a}$



(7)

Energy $E = 8$ (Ex.)

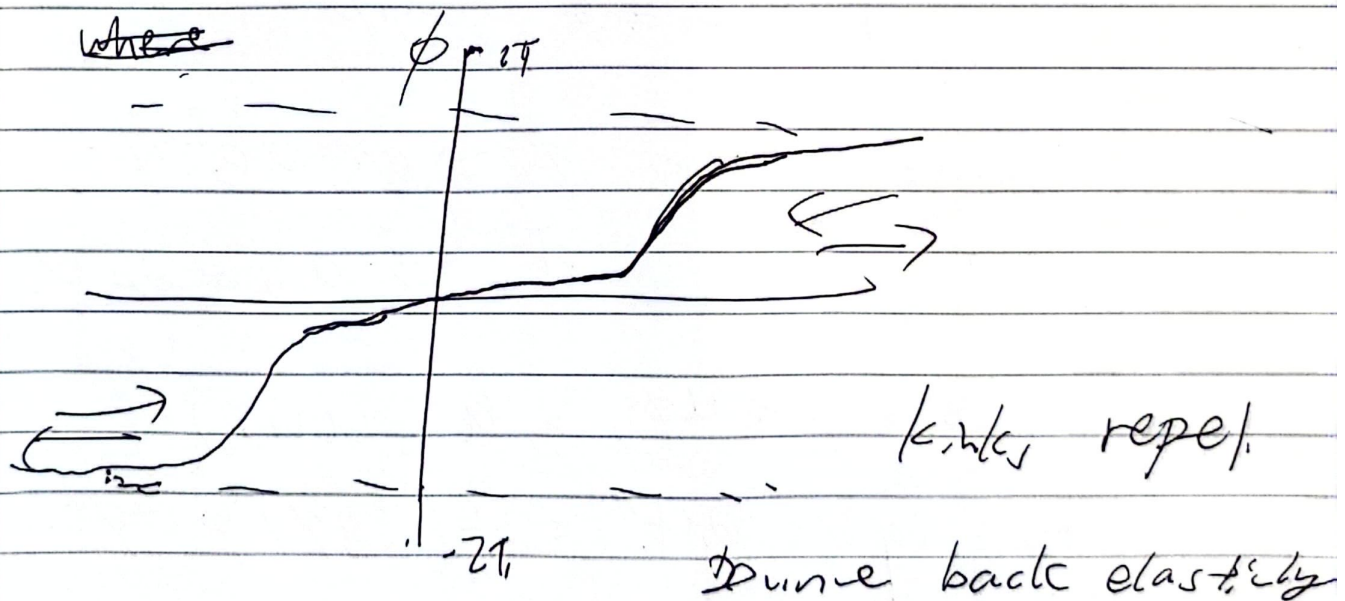
Solitons Lec 4

Sine-Gordon dynamical multi-kink solutions.

These can be found exactly, but algebraically

Example is 2-kink ($Q=2$ but not stable)

$$\phi(x,t) = 4 \tan^{-1} \left(\frac{v \sinh \alpha x}{\cosh \alpha v t} \right)$$



Positions obey a force eq for small v

(~~close~~ closest approach $\sim 2 \log \left(\frac{2}{v} \right)$)

see Top. Solitons p. 116-124

(2)

A model for vortices.Abelian Higgs model 2+1 dim.Complex scalar field ϕ $U(1)$ gauge potential a_μ

Lagrangian density.

$$-\frac{1}{8}(1-\phi^*\phi)^2$$

$$\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2} (D_\mu \phi^*) D^\mu \phi - \frac{1}{8} (1 - \phi^* \phi)^2$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$,

$$D_\mu \phi = \partial_\mu \phi - i a_\mu \phi \quad \text{unit charge}$$

Gauge trans: $\phi \rightarrow e^{i\alpha} \phi$,

$$a_\mu \rightarrow a_\mu + \partial_\mu \alpha$$

or $(\phi \rightarrow u\phi, u \in U(1))$

Full eq. $\left\{ \begin{array}{l} \partial_\mu f^{\mu\nu} = \frac{i}{2} ((D^\nu \phi)^* \phi - \phi^* D^\nu \phi) \\ \text{(current)} \end{array} \right.$

$$D_\mu D^\mu \phi = \frac{\lambda}{2} (1 - \phi^* \phi) \phi$$

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad 2+1 \text{ dim}$$

(3)

Potential energy function (Ginzberg Landau Energy)

$$E = \int d^2x \left\{ \frac{1}{2} B^2 + \frac{1}{2} (D_1 \phi)^* D_1 \phi + \frac{1}{2} (D_2 \phi)^* D_2 \phi + \frac{\lambda}{8} (1 - \phi^* \phi)^2 \right\}$$

Here $B = \partial_1 a_2 - \partial_2 a_1 = \text{magnetic field.}$

Vacuum manifold is circle $|\phi| = 1$

Relevant as description of (planar) superconductors.

$\lambda > 1$ corresponds to Type II behaviour.
(Vortex lattice - Abrikosov)

$\lambda < 1$ I . . .

Critical coupling $\lambda = 1 \Rightarrow$ Bogomolny eqs. (later)

$\rightarrow$ static multi vortices solution

Abelian Gauge Field - Cartesian and Polar in 2-D.

Cartesian components of gauge potential.

$$a_\mu = (a_0, a_1, a_2)$$

(4)

Field tensor $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$

These are components of a coordinate-invariant 1-form, 2-form.

Spatial part ~~$a = a_1 da^1 + a_2 da^2$~~
 $a = a_1 dx^1 + a_2 dx^2$

$$f = da = (\partial_1 a_2 - \partial_2 a_1) dx^1 \wedge dx^2$$

$$= f_{12} dx^1 \wedge dx^2$$

$$\underline{f_{12} = B}$$

Polar coordinates: $x^1 = r \cos \theta$, $x^2 = r \sin \theta$

so $a = a_1 (dr \cos \theta - r \sin \theta d\theta)$

$$+ a_2 (dr \sin \theta + r \cos \theta d\theta)$$

$$= a_r dr + a_\theta d\theta$$

$$\text{in } a_r = a_1 \cos \theta + a_2 \sin \theta = \frac{1}{r} (a_1 x^1 + a_2 x^2)$$

$$a_\theta = -a_1 r \sin \theta + a_2 r \cos \theta = -a_1 x^2 + a_2 x^1$$

$$f = da = (\partial_r a_\theta - \partial_\theta a_r) dr \wedge d\theta$$

$$= f_{r\theta} dr \wedge d\theta$$

$$f_{r\theta} = r f_{12} = r B \quad (\text{since } \cancel{da} \text{ is not})$$

$$dx^1 \wedge dx^2 = r dr \wedge d\theta$$

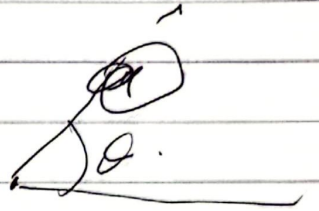
(5)

Vortices and Topology

Typical field $B \rightarrow 0$ $|\phi| \rightarrow 1$ $D\phi \rightarrow 0$
asymptotically (as $|x| \rightarrow \infty$)

Energy limits
of vortices

(2)



(1)

Asymptotic field: Can fix gauge of finite energy field

$$\text{so } \phi_\infty(\theta) = \lim_{r \rightarrow \infty} \phi(r, \theta) \text{ exists.}$$

$$\text{and } |\phi_\infty(\theta)| = 1.$$

$$\text{Write } \phi_\infty(\theta) = e^{i\chi(\theta)} \quad (\chi \text{ coordinates})$$

$\rightarrow \chi(\theta)$ increases by $2\pi N$ as θ increases by 2π for some $N \in \mathbb{Z}$.

N is the winding number of the field

$$\text{: really } \phi_\infty : S^1 \rightarrow S^1_{\text{vacuum manifold}}$$

Solitons lec 5

Vortices and Winding number.

Asymptotic field $\phi_\infty(\theta) = \lim_{r \rightarrow \infty} \phi(r, \theta)$

and $|\phi_\infty(\theta)| = 1$

Write $\phi_\infty(\theta) = e^{i\chi(\theta)}$ (χ continuous)

$\chi(\theta)$ increases by $2\pi N$ as θ increases by 2π

N is winding number of field

Three ~~properties~~ properties

1) N cannot change under continuous deformation of the field including time evolution - hence a topologically conserved quantity - (charges).

2) Asymptotically $D_0 \phi \rightarrow 0$:

$$\partial_0 \phi_\infty - i a_0 \phi_\infty = 0$$

$$\text{so } i(\partial_0 \chi - a_0) = 0$$

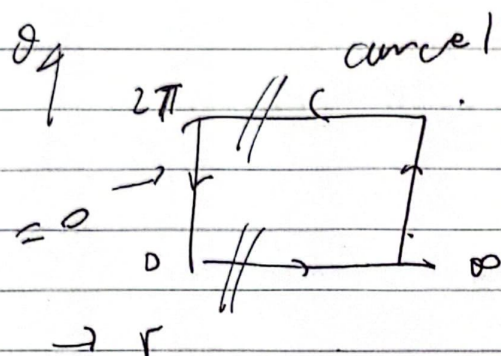
$$a_0 = \partial_0 \chi \text{ on circle at infinity}$$

Gauge field and scalar field are correlated.

(2)

magnetic field

$$\int_{\mathbb{R}^2} B d^2x = \int_{\mathbb{R}^2} f = \int_0^\infty \int_0^{2\pi} f r \, d\theta \, dr.$$



$$\begin{aligned} & \downarrow \\ & 2r dr = dr \\ & 2r dr - 2r dr \end{aligned}$$

by Stokes

$$= \int_0^{2\pi} a \, d\theta \Big|_{r=0}^{r=\infty}.$$

$$\text{using } a \Big|_{r=0}^{r=\infty} = 0.$$

$$= \int_0^{2\pi} 2\pi \chi \, d\theta = \chi(2\pi) - \chi(0) = 2\pi N$$

$\therefore$ Flux is quantized in units of 2π ,

and is proportional to winding number N .

3) N is the number of isolated zeros of ϕ , counted with multiplicity.

$$(x_3)^{n_3}$$

if ~~any~~ any (ϕ) increases by

$$(x_1)^{n_1}$$

$$(x_2)^{n_2}$$

$2\pi n$, around ~~zero~~ zeros x_1 , etc.

(n_1 is the multiplicity of the zero of x_1)

(3)

Then $N = n_1 + n_2 + n_3 + \dots$

If all zeros are simple (generic case), then,

$$N = \# \text{ zeros with mult: } 1$$

$$- \# \text{ zeros with mult: } -1$$

Zeros of ϕ are identified with vortex centres.

The basic vortex: The static field $\mathbf{W}$ of minimised energy with $N=1$ is called basic vortex.

ϕ has a single zero - assume at origin -
and fields are ~~circle~~ ~~only~~ symmetry' circularly

$a_r = 0$	$a_\theta = f(r)$	$\phi = h(r) e^{i\theta}$
$\downarrow$	$\downarrow$	$\downarrow$
gauge choice always possible	$f=0$ at $r=0$ $f=1$ at $r=\infty$	$h=0$ at $r=0$ $h=1$ at $r=\infty$

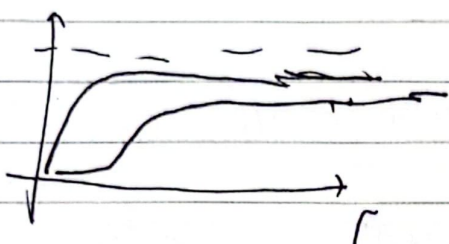
Variational eqns for static fields in polar coords. (EX.)

$$\begin{cases} \frac{d^2 h}{dr^2} + \frac{1}{r} \frac{dh}{dr} - \frac{1}{r^2} (1-f)^2 h + \frac{\Delta}{2} (1-h^2) h = 0 \\ \frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} + (1-f) h^2 = 0 \end{cases}$$

(4)

h grows very slowly (linearly) for large r

f is quadratic for small r .



$$N = -1 \text{ vortex} \quad \left. \begin{array}{l} a_r = 0 \\ a_\theta = -f(r) \\ a_\phi = h(r) e^{i\theta} \end{array} \right\}$$

(There are circularly symmetric vortices

for all N , but they are unstable if $\lambda > 1$

$$q = h_N(r) e^{iN\theta} \quad a_\theta = f_N(r), \quad a_r = 0$$

Boyer's eq for vortices

- Vortex energy bound

Consider the energy of a time-indep. field with winding # N

Assume critical coupling $\lambda = 1$ and set $B = f_{12}$

(5)

$$E = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ B^2 + (D_1 \phi)^* (D_1 \phi) + (D_2 \phi)^* (D_2 \phi) + \frac{1}{4} (1 - \phi^* \phi)^2 \right\} d^2 x.$$

$$= \frac{1}{2} \int_{\mathbb{R}^2} \left\{ B^2 - \frac{1}{2} (1 - \phi^* \phi)^2 + (D_1 \phi + i D_2 \phi)^* (D_1 \phi + i D_2 \phi) \right. \\ \left. + \frac{1}{2} \int_{\mathbb{R}^2} \left\{ B (1 - \phi^* \phi) + i \left((D_2 \phi)^* D_1 \phi - (D_1 \phi)^* D_2 \phi \right) \right\} d^2 x \right\}$$

2nd integral can be expressed as (ex)

$$\frac{1}{2} \int B dx + \frac{1}{2} i \int \left\{ \frac{\partial_2 (\phi^* D_1 \phi)}{\partial_1 (\phi^* D_2 \phi)} - \partial_1 (\phi^* D_2 \phi) \right\} dx$$

using covariant Leibniz rule.

$$\partial_2 (\phi^* D_1 \phi) = (D_2 \phi)^* D_1 \phi + \phi^* D_2 D_1 \phi$$

check cancel out

and using identity $(D_1 D_2 - D_2 D_1) \phi = -i B \phi$

Here, first integral is $\frac{1}{2} (\text{Total flux}) = \pi N$

and second integral is zero by Stokes and $\lim_{r \rightarrow \infty} r D_\mu \phi = 0$

Solitons Lec 6

Bogomolny vortices (abelian Higgs vortices at critical coupling $\lambda=1$)

E reduces to:

$$E = \frac{1}{2} \int_{\mathbb{R}^2} \left[\left(B - \frac{1}{2} (1 - \phi^* \phi) \right)^2 + |D_1 \phi + i D_2 \phi|^2 \right] dx + \pi N$$

for $N > 0$, we get the non-trivial bound

$$E \geq \pi N \quad \text{Bogomolny bound}$$

For $N < 0$, flip the signs to get $E \geq \pi |N|$

The bound is saturated ($E = \pi |N|$) if

$$\left. \begin{aligned} D_1 \phi + i D_2 \phi &= 0 \\ B - \frac{1}{2} (1 - \phi^* \phi) &= 0 \end{aligned} \right\} \text{Bogomolny eqs.}$$

Tambes eq. The Bogo eqs can be reduced to a single non linear, gauge inv. eq.

$$\left. \begin{aligned} \text{Let } \phi &= e^{\frac{1}{2}u + i\chi} \\ \text{Bogo: } \partial_1 \left(\frac{1}{2}u + i\chi \right) - i a_1 \\ &+ i \partial_2 \left(\frac{1}{2}u + i\chi \right) + a_2 = 0 \end{aligned} \right\} \begin{aligned} &u \text{ gauge inv.} \\ &u = \log |\phi|^2 \\ &\chi \text{ real.} \end{aligned}$$

$$\Rightarrow \quad \cancel{a_1 = \frac{1}{2} \partial_2 u + \partial_1 \chi} \quad \begin{aligned} a_1 &= \frac{1}{2} \partial_2 u + \partial_1 \chi \\ a_2 &= -\frac{1}{2} \partial_1 u + \partial_2 \chi \end{aligned}$$

(2)

$$B = \partial_1 a_2 - \partial_2 a_1 = -\frac{1}{2} \nabla^2 u.$$

$$2\text{nd Bogo} \Rightarrow \nabla^2 u - e^u + 1 = 0$$

But u has log singularities at zeros of ϕ

suppose zeros at $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_N$

(Simple multiplicity 1)

$$|\phi| \sim \text{const } |\underline{x} - \underline{x}_i| \text{ near a zero.}$$

$$u \sim 2 \log |\underline{x} - \underline{x}_i|$$

$$\nabla^2 u \sim 4\pi \delta^{(2)}(\underline{x} - \underline{x}_i)$$

$$\Rightarrow \text{Tanabe eq. } \left\{ \begin{aligned} \nabla^2 u - e^u + 1 \\ = 4\pi \sum_{i=1}^N \delta^{(2)}(\underline{x} - \underline{x}_i) \end{aligned} \right.$$

and $u \rightarrow 0$ as $|\underline{x}| \rightarrow \infty$

There is a unique solution for given set of points $\{\underline{x}_i\}$ (Tanabe theorem)

The moduli space $\mathcal{M}_N$ of N -vertex solns is the set of all N -tuples of unordered points in $\mathbb{R}^2$. $\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_N\}$

$$\dim \mathcal{M}_N = 2N$$

(3)

Properties of Bogomolny vortices.

0. Total magnetic flux $2\pi N$,
Total energy πN

1. $B = \frac{1}{2}$ (max value) at vortex centre

1 vortex has flux 2π

$\therefore$ Area of magnetic flux region (for real vortex) $\simeq 4\pi$

2. ~~For~~ Far from vortex centres $u \rightarrow 0$

Taubes eq linearizes to $\nabla^2 u - u = 0$

$\therefore u$ decays exponentially with radius.

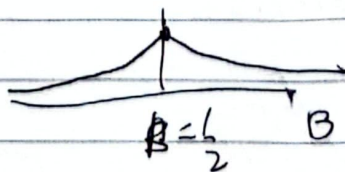
(K_0 Bessel function)

3. u is ~~not~~ everywhere non-positive.

Reason: Where u has a local maximum.

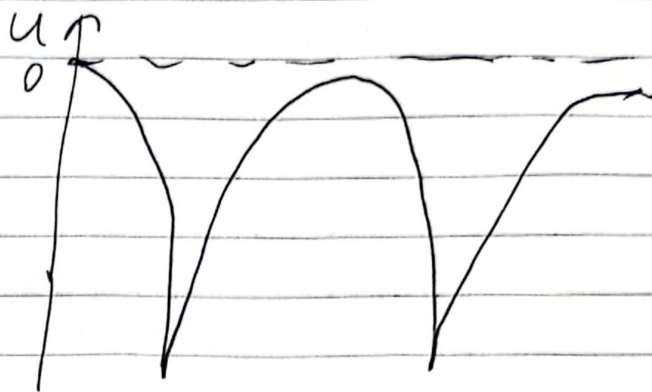
$\nabla^2 u \leq 0$ $\therefore e^u \leq 1$ (by Taubes eq)
^{second derivative}
 $u \leq 0 \Rightarrow |\phi| \leq 1$
 and $B \geq 0$

$\dot{B} = \frac{1}{2}$

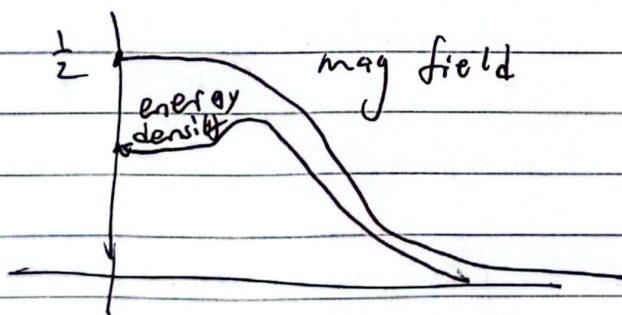
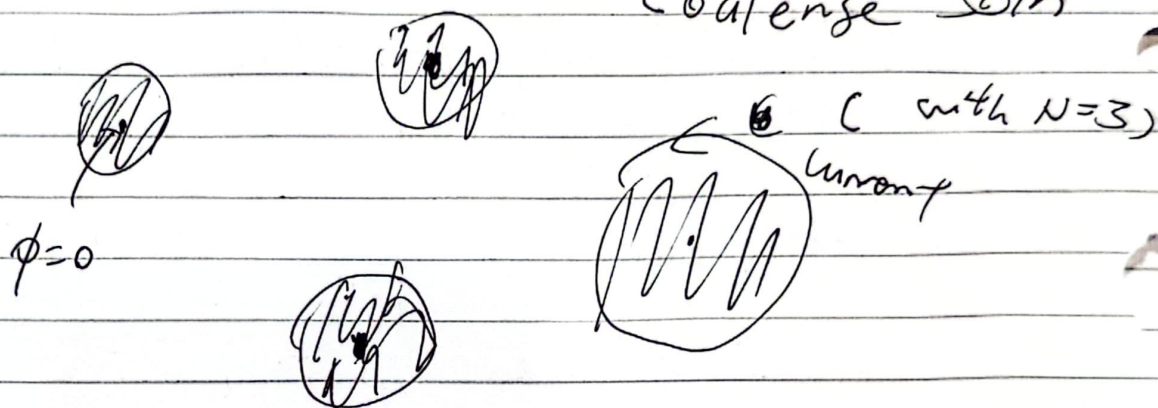


$\therefore u \leq 0$ everywhere

"Graph" of u



typical soln ($N=3$) :



Moduli space M_N - N vortices soln of Bog eq.

N -vortex moduli are $\underline{x}_1, \dots, \underline{x}_N$ modulo permutations

i.e. $M_N = \mathbb{R}^{2N} / S_N \hookrightarrow$ perm. group.

(5)

Better moduli: Introduce complex plane

$$z_r = x_r + i y_r$$

Then define complex polynomial (monic)

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_N)$$

$$= z^N - (\sum_1 + \cdots + \sum_N) z^{N-1} + \cdots$$

$$+ (-1)^N z_1 \cdots z_N$$

Coeffs of this polynomial (sym polys in $\{z_r\}$)

are better moduli: They are SU -invariant
There are N indep coeffs

so $M_N = \mathbb{C}^N$ The rest of $\mathcal{P}$ (vortex locations)
are uniquely determined by the coefficients
of p .

(Surprisingly $M_N = \mathbb{C}^N / S_N = \mathbb{C}^N$)

Solitons Lec 7

Example

$N=2$ vortices. centres at z_1, z_2 .

Sol characterized by

$$p(z) = (z - z_1)(z - z_2)$$

$$= z^2 - (z_1 + z_2)z + z_1 z_2.$$

$z_1 + z_2$ and $z_1 z_2$ are good moduli.

For vortices with centre of mass at origin, then

$$z_1 = z, \quad z_2 = -z$$

so $p(z) = z^2 - b$ where $b = z^2$

vortices at $\pm \sqrt{b}$ Fields evolve smoothly with b , not z

For b real

double zero.
(two vortex solution).



x



$b \gg 0$

$b \geq 0$

$b = 0$



$b \leq 0$



$b \ll 0$



(2)

Field theory gives kinetic energy

$$\frac{1}{2} g_{cb} \dot{b}^2 \text{ and } g \text{ depends only on } |b|.$$

$|b| \dots \Rightarrow$ This is conserved in

Slow motion of vortices ($\because$ total E and (potential = const) are both conserved)

~~ex~~ (no radiation as in fast motion)

— vortices scatter at 90° in head-on

collision. Only possible because of quotient by permutation group S_2 .

— static (Bogomolny) vortices on a curved surface Σ .

set $\lambda=1$, Σ a Riemann surface with metric ~~is~~ either closed (e.g. sphere or higher, no boundary) or open ~~to~~ with boundary at infinite distance. (e.g. hyperbolic plane)

$$|\phi| \rightarrow 1 \text{ on boundary.}$$

Can find (isothermal) coordinates

y^1, y^2 so that metric is locally

$$ds^2 = \Omega(y^1, y^2) (dy^1 + dy^2)$$

(Can also introduce complex coord

$$z = y' + iy'' \quad , \quad ds^2 = \Omega(z, \bar{z}) dz d\bar{z})$$

see Example .

Curvature important (formula later) and also total area A for closed surface
Area element $\Omega dz d\bar{z}$.

Energy (abelian Higgs)

$$E = \frac{1}{2} \int_{\Sigma} \left(\Omega^{-1} B^2 + |D_1 \phi|^2 + |D_2 \phi|^2 + \frac{\Omega}{4} (1 - |\phi|^2)^2 \right) \quad (\lambda=1).$$

E minimized when

$$D_1 \phi + i D_2 \phi = 0 \quad (1)$$

$$B - \frac{\Omega}{2} (1 - |\phi|^2) = 0. \quad (2).$$

and then $E = \pi N$

This uses $\frac{1}{2\pi} \int_{\Sigma} F = \frac{1}{2\pi} \int_{\Sigma} B d^2 y = N$

works for a closed surface, or surface with boundary.

(4)

For closed surface, N is first-Chern number of abelian gauge field.

Taubes eq: set $|\phi|^2 = e^u$, find

$$\nabla^2 u + \Omega e^u + \Omega = 4\pi \sum_{r=1}^N \delta^2(x - \underline{X}_r)$$

Bradlow bound on N

Assume Σ closed, with area A .

Integrate (2)

$$2 \int_{\Sigma} B d^2y + \int_{\Sigma} |\phi|^2 \Omega d^2y$$

$$= \int_{\Sigma} \Omega d^2y = A$$

$$\therefore 4\pi N \leq A \quad \text{as } |\phi|^2, \Omega \geq 0$$

Each vortex has area 4π , Total:

Vortex area $\leq$ Area available

② If $4\pi N = A$, vortices don't dissolve

$$\Rightarrow |\phi| \equiv 0, B = \frac{\Omega}{2}$$

(5)

Baptista's Curvature Eq.

Metric on Σ : $ds^2 = \Omega ((dy^1)^2 + (dy^2)^2)$

(Gaussian).

Gaussian curvature

$$\text{of } \Sigma : K = -\frac{1}{2\Omega} \nabla^2 \log \Omega.$$

$$(\nabla^2 = \partial_1^2 + \partial_2^2)$$

Introduce vortex - modified ~~metric~~
metric on Σ

$$d\tilde{s}^2 = \tilde{\Omega} ((dy^1)^2 + (dy^2)^2)$$

$$\text{with } \tilde{\Omega} = \Omega |\phi|^2 = \Omega e^u.$$

$$\text{Then } \tilde{K} = -\frac{1}{2\tilde{\Omega}} \nabla^2 \log \tilde{\Omega}$$

$$= -\frac{1}{2\Omega e^u} \nabla^2 (\log \Omega + u).$$

use $\rightarrow$ $= \frac{K}{e^u} - \frac{1}{2e^u} (e^u - 1)$

Tambles ignores δ 's.

$$= \frac{1}{e^u} (K + \frac{1}{2}) - \frac{1}{2}$$

$$\Rightarrow (\tilde{K} + \frac{1}{2}) e^u = K + \frac{1}{2}$$

6

Here

more geometrically - multi: by $\Omega(dx_1^2 + dx_2^2)$

$$(Q) \quad (K^2 + \frac{1}{2}) dS^2 = (K + \frac{1}{2}) dS^2.$$

Baptist eg. (geometric interp
of Lambert)

Solitons Lec 8

Baptista Curvature Eq and its use

On $\mathbb{R}^2$ with metric $ds^2 = R dy^2$

Recall $\tilde{R} = R |\phi|^2$, $K =$ Gaussian curvature

The Curvature relation

$$(\tilde{K} + \frac{1}{2}) d\tilde{s}^2 = (K + \frac{1}{2}) ds^2 \quad \text{Baptista eq.}$$

Vortex - modified metric $\tilde{R} dy^2$ preserves

$$(K + \frac{1}{2}) ds^2$$

vortex solns for $K = -\frac{1}{2}$

can construct vortex solns on hyperbolic plane H^2 with

metric (Poincaré disc)

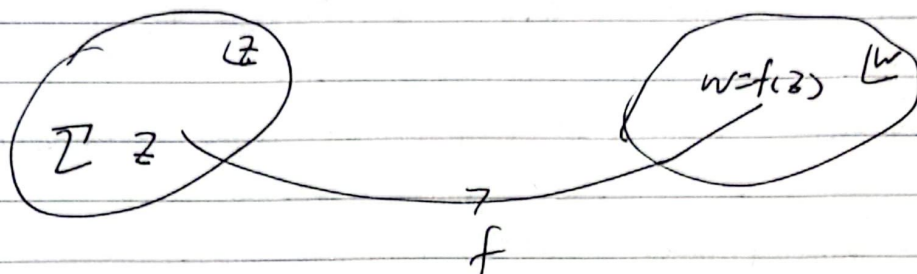
$$ds^2 = \frac{8}{(1 - y_1^2 - y_2^2)^2} (dy_1^2 + dy_2^2) \quad (|y| < 1).$$

This has $K = -\frac{1}{2}$ using complex coord

$$z = y_1 + iy_2, \quad ds^2 = \frac{8 dz d\bar{z}}{(1 - |z|^2)^2}$$

Need to find modified metric with $\tilde{K} = -\frac{1}{2}$ except at isolated point.

use conformal map f between two copies of H^2 (mapping boundary to boundary)



Take metric $d\tilde{s}^2 = \frac{8dw d\bar{w}}{(1-|w|^2)^2}$ and pull back by f (s.t. $w=f(z)$)

$$\text{to get } d\tilde{s}^2 = \frac{8}{(1-|f(z)|^2)^2} \frac{df}{dz} \left(\frac{d\bar{f}}{d\bar{z}} \right) dz d\bar{z}$$

$$= \frac{8}{(1-|f(z)|^2)^2} \left| \frac{df}{dz} \right|^2 dz d\bar{z}$$

This is metric on $\Sigma = H^2$ with $K^2 = -\frac{1}{2}$

as required (mapping locally is just a change of coord)

$$\therefore \text{ can identify } \tilde{\Omega} = \frac{8}{(1-|f(z)|^2)} \left| \frac{df}{dz} \right|^2$$

$$\Omega = \frac{8}{(1-|z|^2)}$$

$$|\phi|^2 = \frac{(1-|z|^2)^2}{(1-|f(z)|^2)^2} \left| \frac{df}{dz} \right|^2$$

satisfies
Lambert eq on Σ

(13)

Can choose gauge

from this can find

$$\phi = \frac{1-|z|^2}{1-f(z)^2} \frac{df}{dz}$$

$$dz, d\bar{z}$$

E.g

$$f(z) = z^n \quad (\text{maps } |z|=1 \text{ to } |f(z)|=1)$$

gives

$$\phi = \frac{1-|z|^2}{1-|z|^{2n}} n z^{n-1}$$

$$= \frac{n z^{n-1}}{1 + |z|^2 + |z|^4 + \dots + |z|^{2n+2}} \quad (*)$$

Has ~~zeros of multiplicity~~ $n-1$ at $z=0$ and

satisfies boundary condition $|\phi|=1$ at $|z|=1$

(*) is a charge $n-1$ vortex on H^2

Most general allowed f is Blaschke product

$$f(z) = \prod_{i=1}^n \left(\frac{z - c_i}{1 - \bar{c}_i z} \right) \quad \text{with } |c_i| < 1$$

$\Rightarrow$ charge $n-1$ vortex on H^2

and vortex centres are where $\frac{df}{dz} = 0$

(4)

Cone Singularity

Metric ds^2 and $\tilde{ds}^2$ have curvature $-\frac{1}{2}$ on H^2 , but ds^2 smooth and $\tilde{ds}^2$ vanishes at vortex centre, where $|\phi| = 0$

$\Rightarrow \tilde{ds}^2$ has conical singularities.

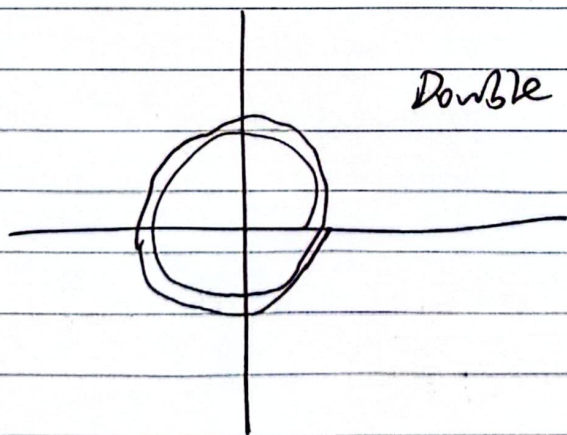
For vortex at origin of multiplicity 1 (for any curvature)

$$\tilde{ds}^2 \cong r^2 (dr^2 + r^2 d\theta^2) \quad 0 \leq \theta \leq 2\pi$$

$$= dV^2 + 4V^2 d\theta^2 \quad \text{for } V = \frac{1}{2}r^2$$

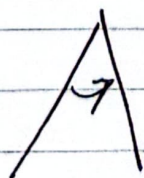
$$= dV^2 + V^2 d\beta^2 \quad \text{for } 0 \leq \beta \leq 4\pi$$

This is a ^{cone} metric with cone angle 4π

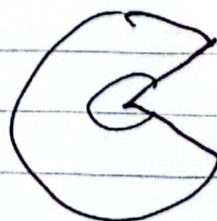


Double cover of flat plane.

Normal cone



=



has cone angle $< 2\pi$

(5)

∴ Baptista metric $\tilde{ds}^2 = \Omega |\phi|^2 (dy^1 + dy^2)$ on Σ
has N cone angle of 4π

Popov vortex

Variant of Bogomolny equation.

$$\begin{cases} D_1 \phi + i D_2 \phi = 0 \\ B + \frac{\Omega}{2} (1 - |\phi|^2) = 0 \end{cases}$$

These give a stationary ~~or~~ point of energy with signs of $|D_2 \phi|^2$ reversed.

Baptista argument $\Rightarrow (\tilde{K} - \frac{1}{2}) \tilde{ds}^2 = (K - \frac{1}{2}) ds^2$

can construct soln on sphere S^2 with

$$ds^2 = \frac{8 dz d\bar{z}}{(1 + |z|^2)^2} \quad \text{and} \quad K = \frac{1}{2} \quad (\text{radius } \sqrt{2})$$

$$\text{soln is } \phi = \frac{1 + |z|^2}{1 + |f(z)|^2} \frac{df}{dz}$$

with $f(z)$ a map ~~on~~ from S^2 to S^2
($z, f(z)$ run over extended complex plane $\mathbb{C} \cup \{\infty\}$)

Globally by acceptable maps $f(z) = \frac{p(z)}{q(z)}$

(ratio of polynomials with no common roots)

Solitons lec 9

(1)

Popov Vortices

Soln is $\phi = \frac{1+|z|^2}{1+(f(z))^2} \frac{df}{dz}$ for $z \rightarrow \infty$

(Relation of z to usual polar coord on $\mathbb{C}^2$ $z = (\tan \frac{1}{2} \theta) e^{i\phi}$).

this is the $\mathbb{C}$ plane + ∞

acceptable are rational map.

$$f(z) = \frac{P(z)}{Q(z)} \text{ with generically } P \text{ and } Q$$

polynomials of some definite degree n with no common roots.

vortex centres where $\frac{df}{dz} = 0$, i.e. where

$$W(z) \equiv \frac{dP}{dz} Q - \frac{dQ}{dz} P = 0$$

The polynomial W is called the Wronskian and (leading $2n-1$ term cancel)

so $2n-2$ degree poly. is W .

$\rightarrow$ has $N = 2n-2$ roots.

N is the vortex (popov) number, always even.

(2)

~~$D\tilde{\epsilon}\phi = 0 \Rightarrow D\tilde{\epsilon}\phi = 0$~~

$D\tilde{\epsilon}\phi = 0 \Rightarrow \text{locally } D\tilde{\epsilon}\phi = 0.$

Section 3

Degree of a Mapping - Rational maps

Degree is a generalisation of winding number

Defined for a map $f: M \rightarrow N$ where M and N are closed connected (compact) manifolds of same dimension d

Suppose ω is a normalised volume form on N (form of highest degree)

i.e. a d -form satisfying $\int_N \omega = 1$.

~~The path~~ ($d = \dim(N)$).

The pull-back ~~f^*~~ $f^*\omega$ is a d -form on M

Defn Degree of f is

$$\deg f = \int_M f^*\omega$$

(3)

E.G. For $f: \mathbb{R}^2 \rightarrow S^2$ and $f(\infty) = p_0 \in S^2$
 $(x, y) \quad (\theta, \phi) \quad \text{mapping } (0, p) \rightarrow (\theta(x, y), \phi(x, y))$

Choose $\omega = \frac{1}{4\pi} \sin \theta d\theta \wedge d\phi$

Then $f^* \omega = \frac{1}{4\pi} \sin \theta(x, y) \left(\frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy \right) \wedge \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \right)$

$= \frac{1}{4\pi} \sin \theta(x, y) \left(\frac{\partial \theta}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial y} \frac{\partial \phi}{\partial x} \right) dx \wedge dy$

and integrate over $\mathbb{R}^2$ to get $\deg f$.

Why is $\deg f$ an integer?

Because $\deg f$ counts preimages of image point $p \in N$.

Key idea is that choice of ω is arbitrary

Suppose $\int_N \omega = 1 = \int_N \tilde{\omega}$

Then $\int_N (\omega - \tilde{\omega}) = 0$

$\omega - \tilde{\omega}$ is an exact form (Integral of ~~that~~ closed form over all available cycles $= 0 \Rightarrow$ form is exact)

$\therefore \omega - \tilde{\omega} = d\alpha$

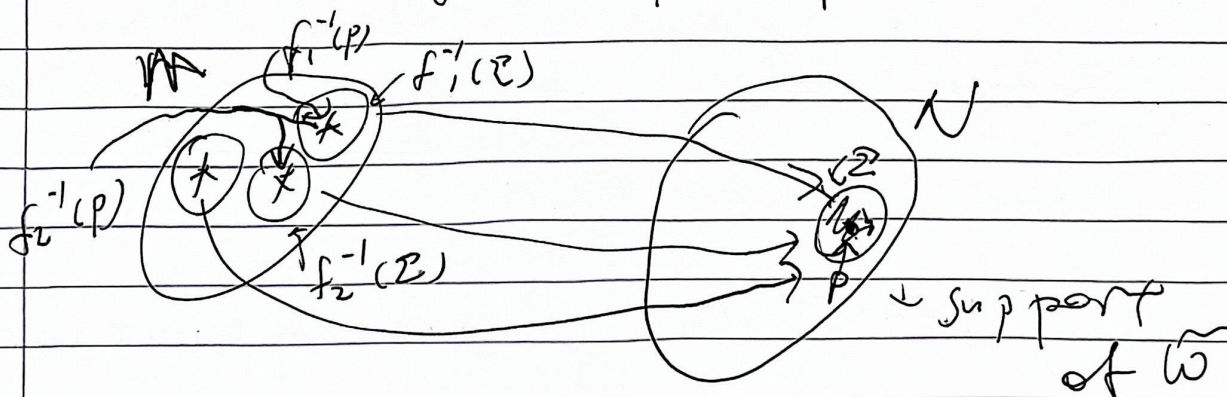
(4)

and $f^*(\omega - \tilde{\omega}) = f^*(d\alpha) = \cancel{df^*\alpha}$
 $\quad \quad \quad = d(f^*\alpha)$
 can be proven.

$\therefore \int_M f^*\omega = \int_M f^*\tilde{\omega}$ as $\int_M$ of exact form = 0.

$$= \deg f.$$

To calculate $\deg f$, choose strongly localized $\tilde{\omega}$ around generic point p



whose preimages of $f^{-1}(p)$ are regular. (f has non-singular matrix of 1st derivatives)

Then $\int_{f_1^{-1}(Z)} f^*\tilde{\omega} = \pm \int_{\Sigma} \tilde{\omega} = \pm \int_N \omega = \pm 1$

(one preimage)

$\tilde{\omega}$ only has support on Σ

because the two integrals are related by a change of variables.

The sign is $\pm$ for f locally.

orientation preserving
orientation reversing.

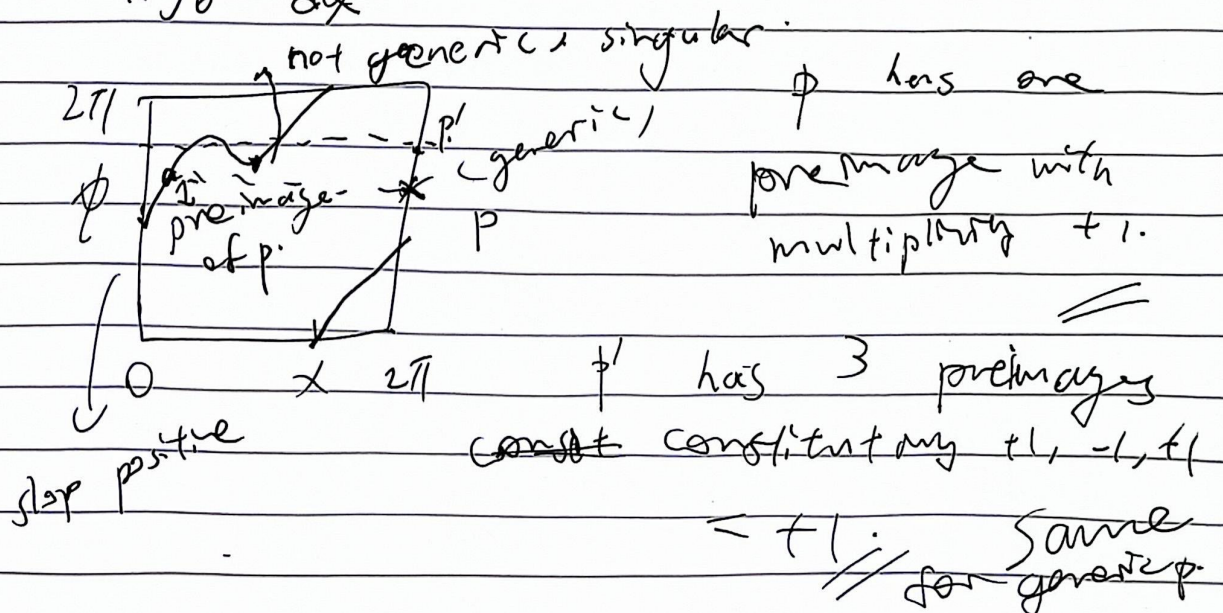
(det of derivative matrix)

$$\therefore \deg f = \int_M f^* \tilde{\omega} = \sum_{\text{preimage of } p} (\pm 1) \in \mathbb{Z}.$$

= # of preimages of (generic) p
counted according to orientation.

E.g. $\phi: S^1 \rightarrow S^1$ with degree 1

as $\frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{dx} dx$



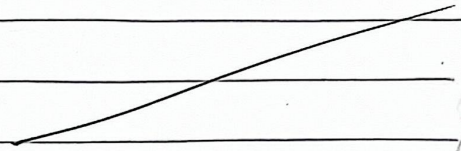
(6)

As ϕ winds once around target circle as α winds around its circle.

$\therefore \text{Degree} = 1$

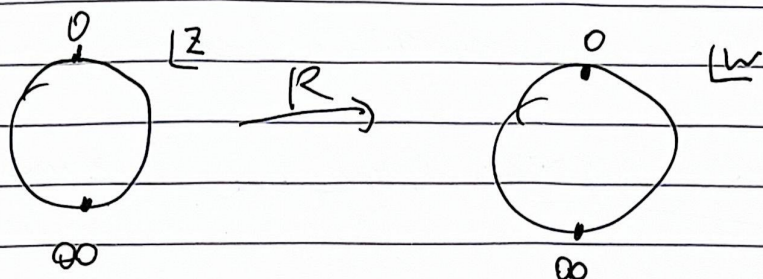
~~def g~~

$\deg f$ is a topological invariant of map f , unchanged by continuous change of w , point p , and f itself.



(1)

Soliton Lec 10

Rational maps consider $R: S^2 \rightarrow S^2$ 

Use Riemann sphere (stereographic) coord

 $z = \tanh(\frac{1}{2}\theta)e^{i\phi}$ on domain S^2 and $\mathbb{C}$.another smooth such coord w on target S^2

$$W = R(z) = \frac{p(z)}{q(z)}, \text{ where } p, q$$

 p, q are polynomial in z , R is a rational map (the common needs) R extends smoothly as a map to the points at infinity ($z = \infty, w = \infty$)As R is holomorphic, it is conformal and preserves orientationThe algebraic degree of R is $\deg_{\text{alg}} R$

$$\text{s.t. } \deg_{\text{alg}} R \equiv \max \{ \deg p, \deg q \}$$

$$\text{E.g. } \deg = 2 \text{ maps } R(z) = \frac{z^2 + az + b}{cz^2 + dz + e}$$

(deg p = highest power of z) (2)

or $R(z) = z^2$, $R(z) = \frac{z}{z^2 - 1}$

~~R~~ all has algebraic degree 2.

The topological degree (the one we're interested in)

deg R equals deg_{alg} R

Why? Convert preimages by solving

$$R(z) = c$$

This becomes $p(z) - c q(z) = 0$ with deg_{alg} R solutions for generic c .

Avoid non-generic c where leading power of z cancels)

$\therefore$ # of preimages of c

$$= \text{deg}_{\text{alg}} R = \text{deg } R.$$

as all preimages contribute +1

Wronskian. useful characterisation of

rational map $R(z) = \frac{p(z)}{q(z)}$ is.

the Wronskian $W(z) = p'(z)q(z) - q'(z)p(z)$

The Wronskian points are zeros of W

(3)

i.e. the sols of $W(z) = 0$

Properties of Wronskian

1) $R' = 0$ where $W = 0$ where energy depend on $|R'|^2$

Wronskian points are holes in the energy density

2) $\deg W = 2 \deg R - 2$ generically.

3) Symmetries of $R \Rightarrow$ Symmetries of W

(concise but not quite true) (more on symmetries later)

4) If W has degree less than $2 \deg R - 2$ then remaining zeros ~~at~~ are at $z = \infty$

(they correspond to the derivatives of map R vanishing at ∞) (check them by change of coord $\xi = \frac{1}{z}$)

e.g. $R(z) = \frac{z^4 + 2\sqrt{3}i z^2 + 1}{z^4 - 2\sqrt{3}i z^2 + 1}$

$\Rightarrow W(z) = -8\sqrt{3}i z (z^4 - 1)$

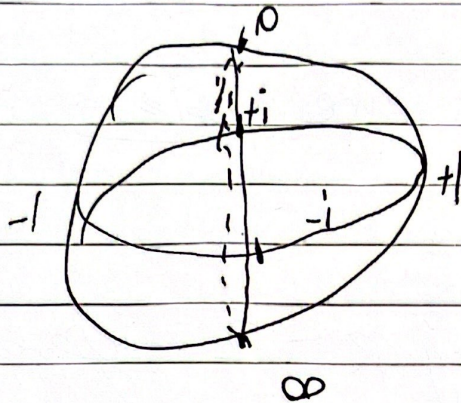
$2(4) - 2 = 6$

Zeros of W at $z = 0, 1, i, -i, -1, \infty$

(14)

(Map has a zero derivative at $z = \infty$
where $R(\infty) = 1$)

These zeros are ~~vertex~~ vertices of
an octahedron.



and $R(z)$ is an octahedrally symmetric
map.

eg (90° rotation) $R(iz) = \frac{1}{R(z)}$

(combines $SU(2)$ Mobius transform).

4. Skyrme model and Skyrmions.

Field theory involves a map (scalar field)

$$U: \mathbb{R}^{3+1} \rightarrow S^3$$

See Skyrme field

$$U(x, t) = \sigma(x, t) \mathbb{I}_2 + i\pi(x, t) \underline{\tau}$$

$\underline{\tau}$ are Pauli matrices.

need $U \in SU(2)$

Requires $\sigma^2 + \underline{\pi} \cdot \underline{\pi} = 1$

so $(\sigma, \underline{\pi}) \in S^3$

Pion fields $(\pi^+) = \pi_1 + i\pi_2$

particles

$(\pi^0) = \pi_3$

$(\pi^-) = \pi_1 - i\pi_2$

Skyrme's idea: a nonlinear theory of
topological
pions can have soliton solutions. representing

baryons. Baryon number $B = (\text{Topological Charge})$ is an integer

Skyrme ModelField $U(x, t) = \sigma(x, t) \mathbb{I}_2 + i \underline{\pi}(x, t) \cdot \underline{\tau}$

$$\in SU(2) \quad \text{so} \quad \sigma^2 + \underline{\pi} \cdot \underline{\pi} = 1$$

Skyrme Lagrangian in massless pion limit

$$\text{i.e. } (\sigma, \underline{\pi}) \in S^3$$

$$L = \int \left[1 - \frac{F_\pi^2}{16} \text{Tr} (\partial_\mu U U^{-1} \partial^\mu U U^{-1}) \right.$$

$$\left. + \frac{1}{32e^2} \text{Tr} ([\partial_\mu U U^{-1}, \partial_\nu U U^{-1}] [\partial^\mu U U^{-1}, \partial^\nu U U^{-1}]) \right]$$

 L invariant under global chiral transf.

$$U \rightarrow O_1 U O_2 \quad O_1, O_2 \in SU(2)$$

$$U^{-1} \rightarrow O_2^{-1} U^{-1} O_1^{-1}$$

(related to $SU(2)_L \times SU(2)_R$ symmetry of QCD with massless u, d quarks only)Need to choose vacuum $U = \mathbb{I}_2$ ($\sigma=1, \underline{\pi}=0$) and impose b.c.

$$U \rightarrow \mathbb{I}_2 \quad \text{as } |x| \rightarrow \infty \quad \text{for any soliton}$$

This choice of vacuum spontaneously breaks chiral symmetry to isospin symmetry

②

$$U \rightarrow O U O^{-1}$$

(Chiral sym really $SO(4)$ $\cong \frac{SU(2) \times SU(2)}{\mathbb{Z}_2}$)

as $O_1 = O_2 = -\mathbb{I}_2$ has ^{no} effect.

iso spin symmetry is $SO(3)$ as $O = -\mathbb{I}_3$ has no effect

Note : L only has terms quadratic in the derivatives.

(pions are goldstone bosons)

Rescale energy / length unit and set

$$R_\mu = (\partial_\mu U) U^{-1} \in L(SU(2)) \quad \begin{array}{l} \text{Lie algebra} \\ \text{of } SU(2) \end{array}$$

= linear comb of Pauli matrices
ie $i\tau$

$$\Rightarrow L = \int \left[-\frac{1}{2} \text{Tr}(R_\mu R^\mu) + \frac{1}{16} \text{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) \right] d^3x$$

Field eq.

$$\partial_\mu \left(R^\mu + \frac{1}{4} [R_\nu, [R^\nu, R^\mu]] \right) = 0$$

Linearised field eq - set $U = \mathbb{I}_2 + i\tau \cdot \pi$

$\Rightarrow$ Massless wave eq $\partial_\mu \partial^\mu \pi = 0$ (ex)

Static fields and energy

Consider static energy

(3)

$$E = \int_{\mathbb{R}^3} \left(\underbrace{-\frac{1}{2} \text{Tr}(R_i R_i)}_{E_2} - \underbrace{\frac{1}{6} \text{Tr}([R_i, R_j][R_i, R_j])}_{E_4} \right) d^3x$$

(i, j = 1, 2, 3).

ie

Any finite-energy stationary (Skyrmion) have

Derrick theorem : $E_2 = E_4$

Pf if $U(x)$ is a solution (stationary ~~point~~ point).

$$\text{or } \tilde{U}(x) = U(\lambda x), \quad \tilde{E} = E$$

$$\tilde{E} = \tilde{E}_2 + \tilde{E}_4 = \frac{1}{\lambda} E_2 + \lambda E_4$$

There is stationary point at $\lambda = 1$

$$\text{So } \frac{d\tilde{E}}{d\lambda} = -\frac{1}{\lambda^2} E_2 + E_4 = 0 \quad \text{at } \lambda = 1$$

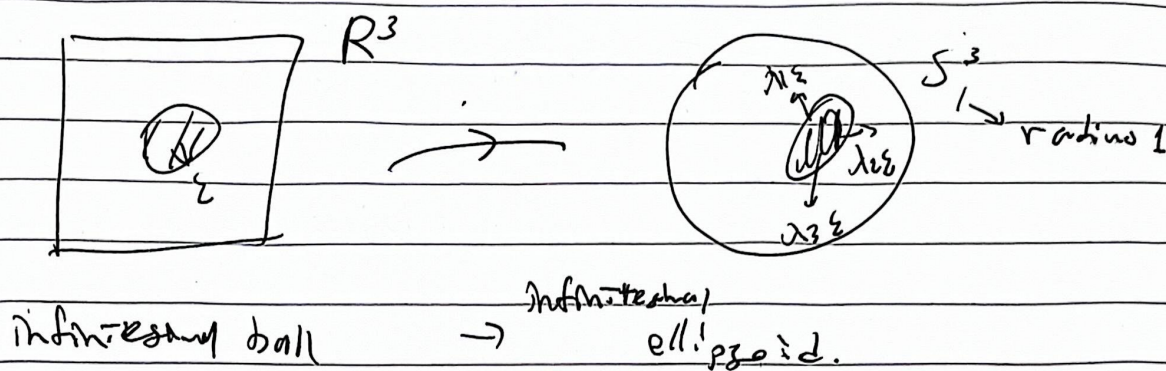
$$\Rightarrow \underline{E_2 = E_4}$$

Consequence - Skyrmion has definite scale size.

Geometric interpretation of energy

U is a mapping from $\mathbb{R}^3$ to S^3

(4)



locally $\mathcal{U}$ maps ball of radius ϵ to ellipsoid with principle axes of length $\lambda_1 \epsilon, \lambda_2 \epsilon, \lambda_3 \epsilon$.

Then

$$E = \int_{R^2} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2) d^3x$$

Remark

Chiral sym $\Rightarrow$ Density indep of location of $\mathcal{U}(x)$ and isotropic.

Commutator structure of sigma term (E_F term) means $\lambda_1^4 + \lambda_2^4 + \lambda_3^4$ are absent

$\lambda_1^2, \lambda_2^2, \lambda_3^2$ are eigenvalues

$$\text{of } -\frac{1}{2} \text{Tr}(R_i R_j) = -\frac{1}{2} \text{Tr}(R_i R_j)$$

(1)

Solitons Lec 12

Topology and Faddeev - Bogomolny Energy Bound.

Complete squares
Can compute ~~sphere~~, and rewrite energy as

$$E = \int_{\mathbb{R}^3} \left[(\lambda_1 - \lambda_2 \lambda_3)^2 + (\lambda_2 - \lambda_1 \lambda_3)^2 + (\lambda_3 - \lambda_1 \lambda_2)^2 \right] d^3x \\ + 6 \int_{\mathbb{R}^3} \lambda_1 \lambda_2 \lambda_3 d^3x$$

Let B be degree of map mapping

$$U: \mathbb{R}^3 \rightarrow S^3 \quad (\text{Boundary } \text{cond. } U \rightarrow \mathbb{R}^2 \text{ as } |x| \rightarrow \infty \Rightarrow \mathbb{R}^3 \text{ can be regarded as } S_\infty^3)$$

B is an integer topological invariant, identified with baryon number

(Skyrme energy in strain eigenvalues)

$$E = \int_{\mathbb{R}^3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2) d^3x$$

Now, $(\lambda_1 \lambda_2 \lambda_3 d^3x)$ is pull-back of standard volume element on $S^3 \subset \mathbb{R}^4$ by U

$\therefore$ 2nd integral is $6B(V(S^3)) = 12\pi^2 B$
(unit 3-sphere has volume ~~$2\pi^2$~~ $2\pi^2$ - Ex)

~~with~~

(2)

$$\therefore E \geq 12\pi^2 B \quad (E \geq 12\pi^2 |B|)$$

with possible equality if $\left\{ \begin{array}{l} \lambda_1 = \lambda_2 = \lambda_3 \\ \lambda_1 = \lambda_2 = \lambda_3 \\ \lambda_3 = \lambda_1, \lambda_2 \end{array} \right.$ all hold

These eqn require $\lambda_1 = \lambda_2 = \lambda_3 = 0$ (Vacuum)
or $\lambda_1 = \lambda_2 = \lambda_3 = 1$ (no strain)

As S^3 and $\mathbb{R}^3$ are geometrically different (not isometric), it is not possible to have a map U with no strain in any extended finite region

For Skyrmons $E > 12\pi^2 B$

Fardeev - Bogomolov ~~Bound~~ Bound.

Skyrmions are the minima of E for each value of B .

We have $\int_{\mathbb{R}^3} \lambda_1 \lambda_2 \lambda_3 d^3x = 2\pi^2 B$

More standard expression for B
Skyrmion solutions $B = -\frac{1}{24\pi^2} \int_{\mathbb{R}^3} \epsilon_{ijk} \text{Tr}(R_i R_j R_k) d^3x$

$B=1$ hedgehog skyrmions

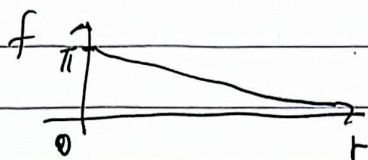
$$R_i = (\sigma \cdot \hat{x}) U$$

$$U(x) = \cos f(r) \mathbb{1} + i \sin f(r) \hat{x} \cdot \underline{\tau}$$

Boundary cond. $U \rightarrow \mathbb{1}$ at ∞

$U \rightarrow -\mathbb{1}$ at origin

(3)



$$E = 4\pi \int_0^\infty \left(f'^2 + 2 \frac{\sin^2 f}{r^2} (1 + f'^2) + \frac{\sin^4 f}{r^4} \right) r^2 dr$$

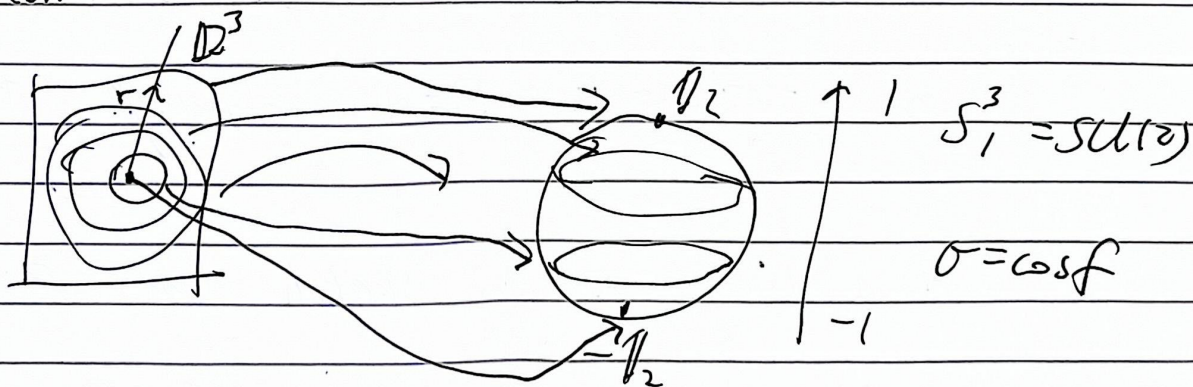
because strain eigenvalues are. $\lambda_1 = -f'$

$$\lambda_2 = \lambda_3 = \frac{\sin f}{r}$$

Baryon number $B = \frac{1}{2\pi^2} \int_0^\infty (-f') \frac{\sin^2 f}{r^2} 4\pi r^2 dr$

$$= \frac{2}{\pi} \int_0^\pi \sin^2 f df = 1$$

Sketch:



(assuming f monotonic)

Eq for f has no explicit soln.

f found numerically ($B=1$)

Skyrme energy $E_1 = 1.232 \times 12\pi^2 > 12\pi^2$

Skyrmions of higher B have to B found

(4)

numerically, but useful is the rational map approximation.

$$V(\underline{x}) = \cos f(r) \frac{\hat{n}}{R(z)} + i \sin f(r) \frac{\hat{n}}{R(z)} \cdot \underline{z}$$

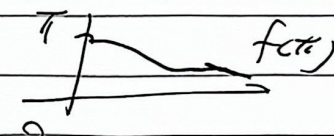
This separates dependence on radial and angular coordinates

$$(r, z = \tan \frac{1}{2} \theta e^{i\phi}) \quad \text{Here}$$

$$\hat{n}_R = \frac{1}{1+|R|^2} (R+R^*, -i(R-R^*), 1-|R|^2)$$

is the unit Cartesian vector corresponding to R (E_x)

$$\text{Plon field } \underline{\pi} = \sin f \hat{n}_R \quad (\hat{\pi} = \hat{n}_R)$$

f similar to $B=1$ hedgehog 

Field V is as in sketch but $R: S^2 \rightarrow S^2$

is ~~related~~ rational map of degree n
($B=1$ case $R(z)=z$)

Baryon number is $B=n$. (as seen by counting preimages of a point on 3-sphere, relies on Boundary cond. and shape of f)

①

Soliton Lec 13

Strain Ex Sheet 3. Q7 should have $4\pi r^2 dr$
~~not $4\pi r^2 dr$~~

Rational map approximation

$$U(x) = (\cos f(r)) \hat{1} + i \sin f(r) \hat{n}_{\text{res}} \cdot \vec{x}$$

Mathematically, the existence of $R: S^2 \rightarrow S^2$
to

$U: S^3 \rightarrow S^1$ is a ~~superior~~ superior

strain eigenvalues are $\lambda_1 = -f'$

$$\lambda_2 = \lambda_3 = \frac{\sin f}{r} \frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right|$$

(uses metrics on S^2 are

$$\frac{dz d\bar{z}}{(1 + |z|^2)^2}, \frac{dR d\bar{R}}{(1 + |R|^2)^2}$$

Total energy for rational map approxi is.

$$E = \int_0^\infty \int f'^2 + 2B \frac{\sin^2 f}{r^2} (1 + f'^2) + \left[\frac{\sin 4f}{r^4} \right] 4\pi r^2 dr$$

from λ_1^2 $\int \frac{\lambda_1^2}{\lambda_2^2} \frac{x_1^2 x_2^2}{x_1^2 x_3^2}$ $\lambda_2^2 \lambda_3^2$

$\mathcal{L}$ is an angular integral depending
(only) on rational map (Ex.)

(2)

We need to optimize choice of $R(z)$ by minimizing I . This is done by numerical variation of coeff. of R

Results are often of high symmetry. Then optimise $f(r)$ by solving ODE

(Details, Nht+PS - 9.5)

We find optimal rat. maps.

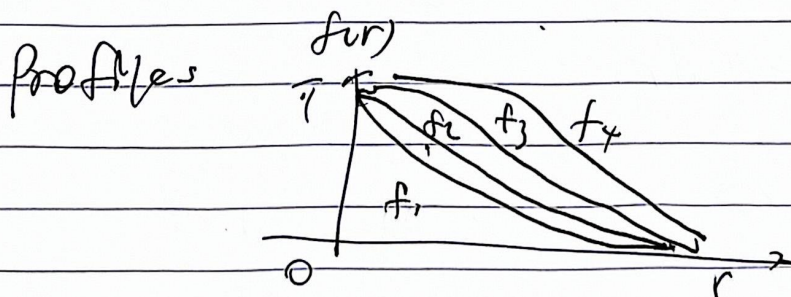
$$R_1(z) = z \quad (\text{cubic})$$

$$B = \leftarrow R_2(z) = z^2 \quad (\text{circularly symm})$$

$$\text{deg } n \text{ of } R \quad R_3(z) = \frac{\sqrt{3}i z^2 - 1}{z^3 - \sqrt{3}i z} \quad (\text{tetrahedral symm})$$

$$R_4(z) = \frac{z^4 + 2\sqrt{3}i z^2 + 1}{z^4 - 2\sqrt{3}i z^2 + 1} \quad (\text{octahedral symm})$$

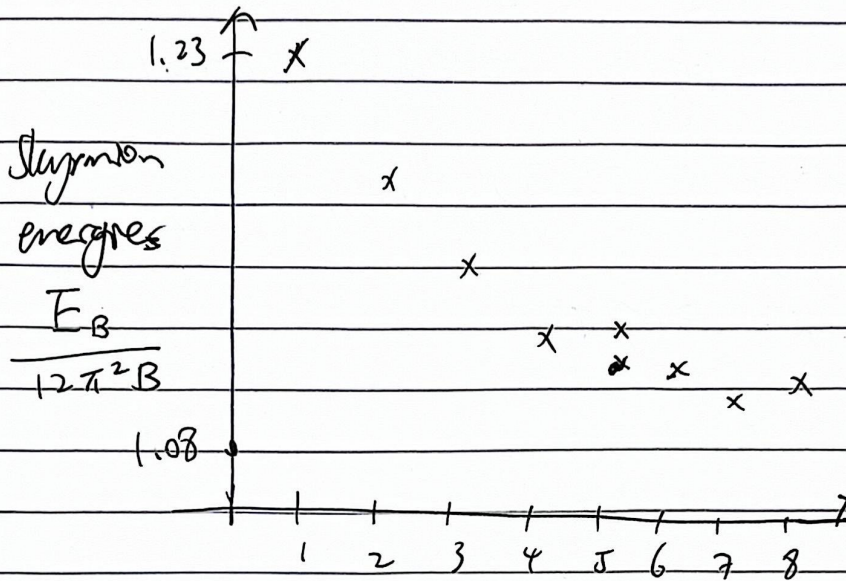
R_7 has icosahedron.



True Skyrmions for $B \leq 8$ have same symmetry and 2-3% lower energy

Holes in baryon and energy densities are at the Wronskian point on $\mathbb{R}$ -sphere (for all r)

$$\# \text{ Wronskian point} = 2B-2$$



These solutions become too hot hollow for large B ($B \geq 8$)

Pion mass term

In original units, can add to Lagrangian a potential term

$$\int -\frac{1}{2} F_\pi^2 m_\pi^2 \text{Tr}(\mathbb{1}_2 - U) d^3x$$

For $U = \sigma \mathbb{1}_2 + i \vec{\pi} \cdot \underline{\tau}$

$$\text{Tr}(\mathbb{1} - U) = 2(1 - \sigma)$$

(4)

$$\text{Tr}(\partial_\mu U) \simeq 2(1 - (1 - \pi \cdot \pi)^{1/2}) \simeq \pi \cdot \pi$$

for small π

Gives pion mass m_π in linearized field eq for pion waves.

pion mass term breaks chiral symmetry but preserves $SO(3)$ isospin sym

$$(m_{\pi^\pm} = m_{\pi^0})$$

Solitons Lec 14

<u>Lectures</u>	Tues 11	MR3
	Wed 11	MR9
	Thurs 11	MR11

Pion mass term prop to $\int \text{Tr} (1 - U) d^3x$

For skyrmions vacuum must be $U = 1$ new,

so $U \rightarrow 1$ as $|x| \rightarrow \infty$

Regions with $U \sim -1$ have high potential energy and are disfavoured. Hollow (rational map) skyrmions are unstable for $B \geq 8$ (for physical m_π)

Skyrmions for larger B more compact, often from $B=4$ cubic units. See fig for $B=12$

There is a skyrmion crystal soln. ($B=\infty$) of low energy per baryon and large B skyrmions are chunks of this

For massless pions $\frac{E}{12\pi^2 B} \approx 1.04$

Crystal is a relaxed version of FCC lattice of $B=1$ skyrmions (occurring in few orientations to produce maximal attraction)

With physical pion mass, it relaxes to a cubic array of $B=4$ skyrmions (with same orientations)

Symmetries of Rational maps and skyrmions

(2)

$R(z)$ has a symmetry if it is invariant under combined rotations of domain/target S^2 (can also have reflection sym)

rotations are realised as $SU(2)$ Mobius transformations of z and R

A rotation $k \in SO(3)$ acts on z -sphere as

$$z \rightarrow R(z) = \frac{\gamma z + \delta}{-\bar{\delta} z + \bar{\gamma}} \quad \text{where } |\gamma|^2 + |\delta|^2 = 1$$

A rot. $M \in SO(3)$ acts R -sphere as

$$R \rightarrow M(R) = \frac{\tau R + \Delta}{-\bar{\Delta} R + \bar{\tau}} \quad \text{where } |\tau|^2 + |\Delta|^2 = 1$$

A map $R(z)$ has symmetry group K if for each $k \in K$ there exist M_k such that

$$R(k(z)) = M_k(R(z)) \quad \forall (z)$$

The pairs (k, M_k) have to combine as in K so

$$M_{k_1} M_{k_2} = M_{k_1 k_2} \quad \forall k_1, k_2 \in K$$

$k \rightarrow M_k$ is a homomorphism

Ex, 1 $R(z) = z^2$

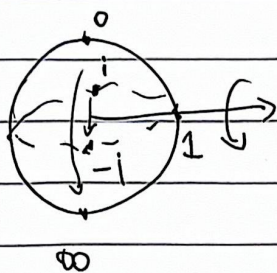
{ Rot. about z -axis by α (on z sphere)
($\gamma = e^{i\alpha}, \delta = 0$)
Iso Rot. about z -axis by 2α (on R sphere)
($\tau = e^{i2\alpha}, \Delta = 0$)

(3)

~~$(\gamma = e^{\frac{1}{2}i\alpha}, \delta = 0)$~~

Find $R(e^{i\alpha} z) = e^{2i\alpha} z^2 = e^{2i\alpha} R(z)$

Another sym $z \rightarrow \frac{1}{z}$ (π -rot about l -axis)



$$R\left(\frac{1}{z}\right) = \frac{1}{z^2} = \frac{1}{R(z)}$$

$$\therefore (\gamma = 0, \delta = i) \quad (\pi = 0, \Delta = i)$$

Symmetry group is $D_{\infty h}$ (toroidal sym)

(Reflection sym $z \rightarrow \bar{z}$, $R(z) = R(\bar{z})$)

Ex 2 $R(z) = \frac{z^4 + 2\sqrt{3}i z^2 + 1}{z^4 - 2\sqrt{3}i z^2 + 1}$

Symmetry $R(z) = \frac{1}{R(z)}$ (k is $\frac{\pi}{2}$ rot
 $m_k \sim \pi$ rot)

There is also a 120° ($\frac{2\pi}{3}$) rot about Cartesian $(1,1,1)$ axis

$$[0 \rightarrow 1 \rightarrow i \rightarrow 0] \quad \text{Ex. 9.}$$

Total symmetry group is cubic group O ,

also reflection symmetry $\Rightarrow O_h$

①

Solitons Lec 15

Wed Lec MR9

Thurs 21st MR3

Symmetries of Skyrmions

Each skyrmion has own point group sym.

$$K \subset \underset{\text{rot.}}{SO(3)} \times \underset{\text{isot.}}{SO(3)}$$

[K realised by ~~180°~~ Möbius transf. in rational map approx.]

K finite except for $B=1$ ($K=SO(3)$)
and $B=2$ ($K=D_{\infty}$)

Elements of form $(r, M(r))$ with $r \rightarrow M(r)$
@ a homomorphism i.e. $M(r_2 r_1) = M(r_2) M(r_1)$

Action of rotation r equivalent to action
of isorotation $M(r)$ on the ~~the~~ Skyrme
field. Skyrmion invariant under ~~rot~~ r
and isotrot with $M(r)^{-1}$.

E.g. $B=1$ hedgehog $\vec{T} = \sin f(r) \hat{x}$
so rot. r equivalent to isotrot $M(r) = r$
for all $r \in SO(3)$

Skyrmion Quantisation

Field config space (Baryon ~~not~~ number
 B)

$$G_B = \text{Maps}_B (\mathbb{R}^3 \rightarrow SU(2))$$

$\uparrow$
deg = B

$$= \text{Maps}_B (S_\infty^3 \rightarrow S_1^3)$$

$$\stackrel{\text{topologically}}{\simeq} \text{Maps}_0 (S_\infty^3 \rightarrow S_1^3)$$

related by multiplying
by a fixed field config
with baryon number B
(1 to 1 mapping)

vacuum sector
no skyrmion

G_B is connected but not simply connected.
(non-contractible loops) as

$$\pi_1 (\text{Maps}_0 (S^3 \rightarrow S^3)) = \pi_4 (S^3) = \mathbb{Z}_2$$

$\downarrow$ 1st homotopy group $\uparrow$ 4th homotopy group

wavefunction Ψ are ~~single~~ single-valued
on double cover of G_B

Fermionic quantisation (Skyrme) Ψ changes

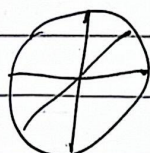
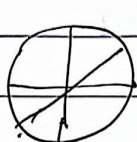
sign on going round a non-contractible
loop in G_B (Baryons are fermions)

Consequences : (1) Ψ changes sign when $B=1$
Skyrmion rotates by 2π . So
spin is half-integer.
Spring in a field theory of
pions

(ii) Φ changes sign when time $B=1$
 Skyrmions • exchanged. (Finkelstein -
 Rubinfeld)

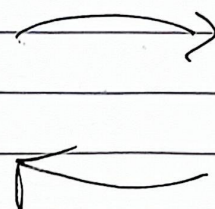
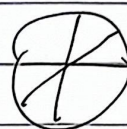
(i)

(ii')



2π

=



(iii) Rotation or Isorotation by 2π is
 contractible for B even $\Rightarrow$ integer spin
 and isospin
 non-contractible for B odd $\Rightarrow$ half-integer
 spin/isospin.

Rigid Body quantisation

Consider Skyrmion collective coordinates (moduli)
 of translations, rotations and isorotations.

Translations $\Rightarrow$ wavefunction of Skyrmion
 has definite momentum $\underline{P}$

Focus on rotation and isorotation.

Wavefunction are combinations of

3

$\vec{J}^2$

white



Red

Black

$$\Psi = |J \ L_3 \ J_3\rangle \otimes |I \ K_3 \ I_3\rangle$$

total
angular
mom.

3rd
comp
w.r.t
body
fixed
axis

3rd
cpt
in
space

total
isospin

3rd
cpt
of
body
fixed
isospin

3rd cpt
of isospin

Realise these wavefunctions using Wigner D-functions of Euler angles.

Bodyfixed L_3, K_3 constrained by Skyrman symmetry group. That ~~limits~~ constrains J, I .

J_3, I_3 not constrained within usual range.

$$-I \leq I_3 \leq I \quad -J \leq J_3 \leq J$$

Sditans Lec 16

Examples of Sleyrmian Quantisation

Ex. 1 $B=1$ hedgehog

$SO(3)$ - symmetries with $M(r) = r$

$$r \in SO(3) \quad \therefore e^{i\alpha \hat{n} \cdot \underline{L}} \leq e^{i\alpha \hat{n} \cdot \underline{K}} |\Psi\rangle = |\Psi\rangle$$

for any angle α , any and rotation axis $\hat{n}$

$$\Rightarrow (\underline{L} + \underline{K}) |\Psi\rangle = 0 \quad (\Psi \text{ has grand spin } 0 \\ = \text{spin} + \text{isospin})$$

$$\Rightarrow J = I \quad (\underline{J}^2 = \underline{L}^2, \underline{I}^2 = \underline{K}^2)$$

$$e^{i2\pi \hat{n} \cdot \underline{L}} |\Psi\rangle = -|\Psi\rangle \quad (B=1 \text{ FR condition})$$

Allowed states $\begin{pmatrix} p \\ n \end{pmatrix}^{\uparrow\downarrow}$ with $J = \frac{1}{2} \quad I = \frac{1}{2}$
nucleons

p has $I_3 = \frac{1}{2}$ n has $I_3 = -\frac{1}{2}$

Also $\begin{pmatrix} \Delta^{++} \\ \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix}^{\uparrow\downarrow}$ with $J = \frac{3}{2} \quad I = \frac{3}{2}$
Delta resonances $J \leftrightarrow N + \pi$

$$M_{\text{mes}} M_{\text{nuc}} = E_{B=1} + \frac{3}{4} \frac{\hbar^2}{2u}$$

$J(J+1)$ $\uparrow$ mom of inertia
for $J = \frac{1}{2}$

$$M_{\Delta} = E_{B=1} + \frac{15}{4} \frac{\hbar^2}{2u}$$

Ex. 2 $B=2$ tons.

Rat. map $R(z) = z^2$ has symmetries

$$R(e^{i\alpha} z) = e^{2i\alpha} R(z), \quad R\left(\frac{1}{z}\right) = \frac{1}{R(z)}$$

$$\Rightarrow e^{i\alpha L_3} e^{i2\alpha K_3} |\mathbb{E}\rangle = |\mathbb{E}\rangle$$

for any α

$$e^{i\pi L_3} e^{i\pi K_3} |\mathbb{E}\rangle = -|\mathbb{E}\rangle$$

(For PR sign - see Ex 08)

Allowed states require $(L_3 + 2K_3) |\mathbb{E}\rangle = 0$

$$L_3 + 2K_3 = 0 \quad (\text{as eigenvalues})$$

Provided $J < 2$ $L_3 = K_3 = 0$

Allowed states include

$$|1, 0\rangle \otimes |0, 0\rangle \quad \text{Deuteron}$$

$$J=1 \quad L_3$$

$$I=0 \quad K_3$$

Isospin 0
bound state
of p, n

$$|0, 0\rangle \otimes |1, 0\rangle \quad \text{Dinucleon}$$

$$J \quad L_3$$

$$I \quad K_3$$

Isospin 1
resonance of
 nn, pp or np

Note $e^{i\pi L_3} |J, L_3\rangle$

$$= (-1)^{J+L_3} |J, -L_3\rangle$$

So $J=0, I=0$ state not allowed.

Ex.3 $B=4$ cubic skyrmion

More restrictive constraints FR signs are all +1

For $I=0$, have $J=0, 4$
(no $J=2$ state)

lowest $I=1$ state has $J=2$

$I=0, J=0$ state is α -particle (ppnn)

$I=1, J=2$ state is pnnn resonance

FR signs for ~~B~~ Rat. Map Skyrmion (Krusch)

Each symmetry gives a quantum constraint

$$e^{i\theta_2 \hat{n}_2 \cdot L} e^{i\theta_1 \hat{n}_1 \cdot K} |\Psi\rangle = \chi_{FR} |\Psi\rangle$$

$$\text{Then } \chi_{FR} = (-1)^{\frac{B}{2\pi} (B\theta_2 - \theta_1)}$$

Proof, follow ~~math~~ ~~math~~ motion of zeros and poles of rat. map.

$$\text{E.G. } B=2 \quad \theta_2 = \alpha, \quad \theta_1 = 2\alpha \quad \chi_{FR} = 1$$

$$B=2 \quad \theta_2 = \pi, \quad \theta_1 = \pi \quad \chi_{FR} = -1$$

$$B=4 \quad \theta_2 = \frac{\pi}{2}, \quad \theta_1 = \pi \quad \chi_{FR} = 1$$

$$B=4 \quad \theta_2 = \frac{2\pi}{3}, \quad \theta_1 = \frac{2\pi}{3} \quad \chi_{FR} = 1$$

Large B skyrmions often have an approx. by multi layer rat. maps and Krusch formula still useful.

Summary:

1. Skyrme model complicated as a field theory, but skyrmions capture important degree of freedom

⇒ Spin - isospin, some vibrational

2 consistent with basic QCD

Baryon number, Pions, Fermionic
p, n.

3. A few cases work well, e.g.

^{12}C ^{16}O many nuclei still ^{to be} tackled

e.g. ^9Be ^{14}C ^{14}N

4 Few parameters, but not very accurate binding energies.

etc Including p-mesons.

(Sutiaff, Vaya)

I, More work on large B,

Columb. effects